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Closure spaces that are not uniquely generated

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Abstract

Because antimatroid closure spaces satisfy the anti-exchange axiom, it is easy to show that they are uniquely generated. That is, the minimal set of elements determining a closed set is unique. A prime example is a discrete convex geometry in Euclidean space where closed sets are uniquely generated by their extreme points. But, many of the geometries arising in computer science, e.g. the world wide web or rectilinear VLSI layouts are not uniquely generated. Nevertheless, these closure spaces still illustrate a number of fundamental antimatroid properties which we demonstrate in this paper. In particular, we examine both a pseudo-convexity operator and the Galois closure of formal concept analysis. In the latter case, we show how these principles can be used to automatically convert a formal concept lattice into a system of implications.

Keywords: Galois closure; Antimatroid; Convex; Concept lattice; Disjunctive implication

1. Overview

Matroids and antimatroids can be studied either in terms of a family $\mathscr{F}$ of *feasible* sets and a shelling operator σ [1,11], or in terms of a collection $\mathscr{C}$ of closed sets and a closure operator φ [3,14]. There exists a considerable amount of confusion, and an equally great richness, because these are two distinct approaches to precisely the same concepts. Given an antimatroid universe, $\mathbf{U}$, every feasible set $F \in \mathscr{F}$ is the complement of a closed set

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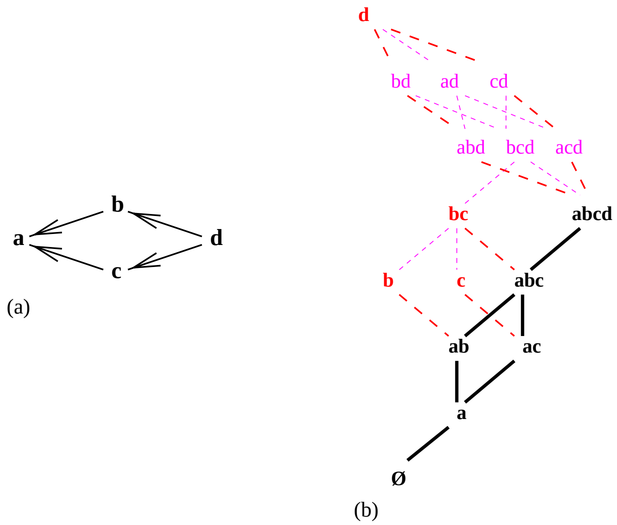


Fig. 1. Closure defined on a graph G (a) and its lattice, $\mathscr{L}$, (b).

$C \in \mathscr{C}$, that is $F = \mathbf{U} - C$ and conversely. In this paper we will choose to emphasize the "closure" approach.

Similarly, there exist many well-known results concerning $\mathscr{F}$ and/or $\mathscr{C}$. For example, many individuals have observed that $\mathscr{C}$, partially ordered by inclusion, is a lower semimodular lattice, $\mathscr{L}_{\mathscr{C}}$ [12], and Edelman demonstrated the stronger result that $\mathscr{C}$ is meet-distributive [2]. Less is known about those sets of $\mathbf{U}$ that are in neither $\mathscr{F}$ nor $\mathscr{C}$. This paper concentrates on those such subsets, together with closure systems which may not quite be "antimatroid".

Let $(\mathbf{U}, \varphi)$ be a closure system satisfying the usual closure axioms, that is $\forall X, Y \subseteq \mathbf{U}$, (a) $X \subseteq X.\varphi$, (b) $X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$, and (c) $X.\varphi.\varphi = X.\varphi$. $(\mathbf{U}, \varphi)$ is called a *matroid* if it satisfies the exchange axiom, that is, if $p, q \notin X.\varphi$ and $q \in (X \cup \{p\}).\varphi$, then $p \in (X \cup \{q\}).\varphi$. On the other hand, $(\mathbf{U}, \varphi)$ is called an *antimatroid* if it satisfies the anti-exchange axiom, that is, if $p, q \notin X.\varphi$ and $q \in (X \cup \{p\}).\varphi$, then $p \notin (X \cup \{q\}).\varphi$. It is not hard to show that antimatroid closure spaces are *uniquely* generated, in the sense that every closed set Z has a unique, minimal subset X with closure $X.\varphi = Z$. Such unique generating (or basis, or irreducible) sets have been denoted by $X.\gamma$. When the closure φ is antimatroid, there is a tight relationship between any closed set and its generators, which is expressed by

Theorem 1.1 (*Fundamental covering theorem*). *Let $Z \in \mathscr{C}$ be any closed set. $Z - \{p\}$ is closed if and only if $p \in Z.\gamma$.*

An ordering, $X \leqslant_{\varphi} Y$ if $Y \cap X.\varphi \subseteq X \subseteq Y.\varphi$, on all subsets $X, Y \subseteq \mathbf{U}$, was introduced in [14] which also introduced graphic representations such as in Fig. 1. When closure is taken to be $Y.\varphi = \{x | (\exists y \in Y)[x \leqslant y]\}$, the lattice of Fig. 1(b) illustrates the ordering of all subsets of (a), the graph G. The closed sets, $C \in \mathscr{C}$ are those connected by solid lines denoting subset inclusion. Apparently, inclusion relationships exist between the non-closed sets that mirror those of the closed sets; that is each interval $[Z.\varphi, Z.\gamma]$ is a Boolean lattice with the property that, if $X.\varphi \leqslant_{\varphi} Z.\varphi$ then $[X.\varphi, X.\gamma]$ is isomorphically embedded in $[Z.\varphi, Z.\gamma]$ by $\sigma : Y \to Y \cup \Delta$, where $\Delta = Z.\varphi - X.\varphi$. This was proven in [14]. Does this kind of replicated structure exist if the closure operator is not antimatroid?

2. Pseudo-convexity

Ideal operators, such as φ illustrated in Fig. 1, and discrete convexity operators such as those developed in [3,4,8,9,13] are a rich source of antimatroid closure spaces. But, the authors know of no well-defined convexity operator over a discrete pixel space. However, a pseudo-convex operator, based on alternate expansion and contraction was developed in [16] and subsequently used to implement a variety of digital image processing operators.

One can extend expansion–contraction operation to undirected graphs $G = (N, E)$, where N is a set of nodes and E a set of edges. For any $Y \subseteq N$ we let $Y.\eta$ denote the open neighborhood of Y, that is $Y.\eta = \{z \notin Y | \exists y \in Y \wedge (y, z) \in E\}$, and let $Y.\bar{\eta}$ denote the *closed neighborhood*, or $Y.\eta \cup Y$. By the *neighborhood closure*, φ_η we mean the set $Y.\varphi_\eta = \{z | z.\bar{\eta} \subseteq Y.\bar{\eta}\}$. Notice that this closure concept precisely captures the process of expansion and contraction of the preceding section because $z \in Y.\varphi_\eta$ if and only if each neighbor of z is a neighbor of Y, and hence “filled” when Y is expanded.

Lemma 1. *φ_η is a closure operator.*

Proof. It is apparent that $Y \subseteq Y.\varphi_\eta$ and $X \subset Y$ implies $X.\varphi_\eta \subseteq Y.\varphi_\eta$. Only idempotency is questionable because, in general, $Y.\bar{\eta} \subset Y.\bar{\eta}.\bar{\eta}$.

Let $y \in Y.\varphi_\eta.\varphi_\eta$ and suppose $y \notin Y.\varphi_\eta$. The latter implies $\exists z \in y.\bar{\eta}$ such that $z \notin Y.\bar{\eta}$. But, $y \in Y.\varphi_\eta.\bar{\eta}$ requires that $\exists y' \in Y.\varphi_\eta$ such that $z \in y'.\bar{\eta}$. However, this implies $y' \notin Y.\varphi_\eta$ (for the same reason that $y \notin Y.\varphi_\eta$) and contradiction. So $y \in Y.\varphi_\eta$ and $Y.\varphi_\eta.\varphi_\eta \subseteq Y.\varphi_\eta$. $\square$

In Fig. 2, we have a small 8 element graph with 26 subsets closed under φ_η as shown in Fig. 2(b). Clearly, φ_η is not a uniquely generated closure. We see that $\{eg, eh\}$ are minimal generators for *egh*.[2] Similarly, $\{ce, bde\}$ minimally generate *abcde*. There are 14 minimal generators of $\mathbf{U} = abcdefgh$. We have only sketched in a few of the 171 subsets whose closure is **U** to suggest this structure. The subsets of **U**, partially ordered by $\leqslant_\varphi$, are *not* a lower semi-modular lattice.

But, many of the important properties of closure lattices still hold. We observe that each of the structures $[X.\varphi_\eta, X.\gamma]$[3] is isomorphically replicated in $[Z.\varphi_\eta, Z.\gamma]$ by $\sigma : Y \to Y \cup \Delta$, $X.\varphi_\eta \leqslant_\varphi Y \leqslant_\varphi X.\gamma$, where $\Delta = Z.\varphi_\eta - X.\varphi_\eta$.

[2] For simplicity, we will enumerate elements *egh* to denote the set rather than use the more correct $\{egh\}$.

[3] Because φ_η is not uniquely generated, these intervals are not Boolean algebras.

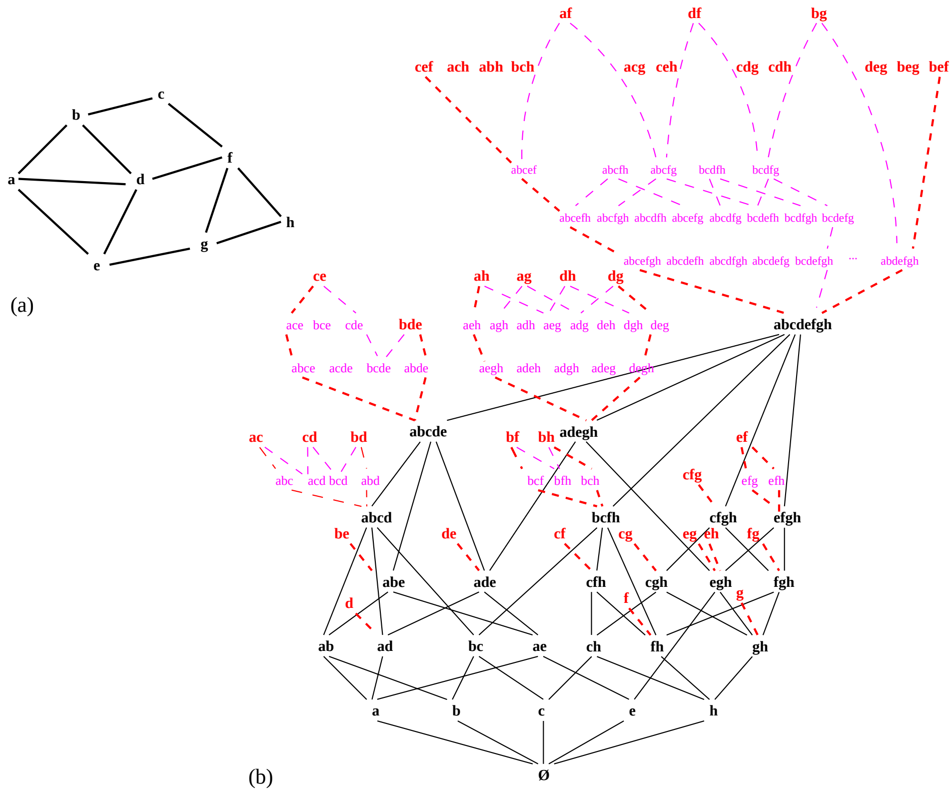


Fig. 2. Pseudo-convexity operator.

We should note that the concept of dominance is one of the oldest, and still one of the most vital, themes in abstract graph theory [7]. Our concept of pseudo-convexity subsumes domination theory, because a subset Y is said to dominate $G=(N,E)$ if $Y.\bar{\eta}=N$. If Y is a minimal set with this property, then Y is a familiar *domination set*, that is, Y is a generator for **U**. Similarly, we observe that [10] has employed an expansion–contraction approach to web search that has been quite successful. Closure concepts overlap many research areas.

3. Generalized closures

In this section we generalize the development of antimatroid closure spaces found in [14,15]. First, since $Y.\gamma$ denotes a minimal generator of $Y.\varphi$, when the closure operator φ is not uniquely generated, we let $Y.\Gamma=\{Y.\gamma\}$ denote the *set* of all minimal generators of Y. Let Z be closed and let $X_i\subseteq Z$ denote maximal closed subsets. If $\mathscr{S}=(\mathbf{U},\varphi)$ is an antimatroid closure space, then Theorem 1.1 shows that $Z-X_i=\{p_i\}\in Z.\gamma$. In our generalization to closure systems which need not be antimatroid we allow $Z-X_i$ to be an arbitrary set Δ_i, which we call a *face* of Z. The collection $\beta_Z=\{\Delta_i\}=\{Z-X_i\ :\ X_i\subset Z,\ X_i$ maximal, closed} we call the *boundary* of Z. In the closure space of Fig. 1, the boundary $\mathbf{U}.\beta$ of the entire space $\mathbf{U}$ is $\{d\}$. The boundary of the pseudo-convex space shown in Fig. 2 is more interesting. Here, $\mathbf{U}.\beta = \{fgh, bcf, adeg, abde, abcd\}$.

For non-uniquely generated closure spaces, Theorem 1.1 is no longer valid; it must be generalized. To do this we use the concept of blockers. Let $\mathscr{F}$ be any family of sets. A set B is said to be a *blocker* for $\mathscr{F}$ if $B \notin \mathscr{F}$ and $\forall X \in \mathscr{F}, B \cap X \neq \emptyset$.

Theorem 2 (*Generalized fundamental covering theorem*). *Let Z be closed with respect to φ and let $Z.\Gamma = \{Z.\gamma\}$ be its family of minimal generators.*

(a) *If $X \subset Z$ and X is closed, then $Z - X$ is a blocker of $Z.\Gamma$.*
(b) *If B is a minimal blocker of $Z.\Gamma$, then $Z - B$ is closed.*
(c) *Z. covers X in $\mathscr{L}_\varphi$ if and only if $Z - X$ is a minimal blocker of $Z.\Gamma$.*

Proof. (a) Let $Z.\gamma \in Z.\Gamma$ and suppose $Z.\gamma \cap (Z - X) = \emptyset$. Then, since $Z.\gamma \subseteq Z$, $Z.\gamma \subseteq X$. But, $Z.\gamma.\varphi = Z$ and thus $Z \subseteq Z.\gamma.\varphi \subseteq X.\varphi = X$, a contradiction.

(b) Let $Y = (Z - B).\varphi$. Then $Y \subset Z.\varphi = Z$. If $Y = Z$, then $Z - B$ is a generating set for Z, so it contains some minimal generating set $Z.\gamma$. Now, $Z.\gamma \subseteq Z - B$ implying $Z.\gamma \cap B = \emptyset$, contradicting assumption that B is a blocker. So $Y \neq Z$.

Since Y is closed and $Y \subset Z$, by (a) $Z - Y$ is a blocker of $Z.\Gamma$. Because $Z - Y$ is a blocker, and because $Z - Y = Z - (Z - B).\varphi \subseteq Z - (Z - B) = B$, and because B is a minimal blocker, we have $B = Z - Y$. Thus $Y = Z - B$, and since Y is closed, $Z - B$ must be as well.

(c) Readily follows from (a) and (b). If Z covers X in $\mathscr{L}_\varphi$, then $Z - X$ is a minimal blocker of $Z.\Gamma = \{Z.\gamma\}$; and if B is a minimal blocker of $Z.\Gamma$, then $X = Z - B$ is closed and Z covers X. $\square$

That is, we may pick an element from each of the generating sets (subject to the constraint that the elements are distinct and do not themselves constitute a generating set). Deletion of such a set $\Delta = \cup Y_i$ from Z will yield another closed set that will be covered by Z with respect to φ. Observe, in Fig. 2 that from the 4 generating sets of $\{adegh\}.\gamma = \{ah, ag, dh, dg\}$ one may choose $\Delta_1 = ad$ or $\Delta_2 = gh$; but no others. Hence, these constitute the bounding faces of the subset *adegh*. Each face is a minimal blocker of the set $Z.\Gamma$ of minimal generators; and conversely each generator $Z.\gamma \in Z.\Gamma$ is a minimal blocker of the set $Z.\beta$ of faces of Z.

A lattice is *meet distributive* if every element Z that covers the set $\{Y_1, Y_2, \ldots, Y_n\}$ of elements is the *supremum* of a distributive sublattice whose *infinium* is $Y_1 \wedge Y_2 \wedge \cdots \wedge Y_n$. When the closure is antimatroid, meet distributivity is an important characteristic of the closed set lattice, $\mathscr{L}_\mathscr{C}$ [2]. The lattice $\mathscr{L}_\mathscr{C}$ of Fig. 2 is not meet distributive. But, many closure spaces, or portions of them, are meet distributive even if they are not antimatroid. Antimatroid closure is a sufficient condition for meet distributivity, but not necessary.

Theorem 3. *Let $\mathscr{F}$ be an anti-chain of sets with $\mathscr{F}.U$ its closure with respect to union. Then the following are equivalent*:

(a) *$\mathscr{F}.U$ is Boolean*;

(b) *$\mathscr{F}.U$ is distributive*;
(c) *no member of $\mathscr{F}$ is covered by other members of $\mathscr{F}$.*

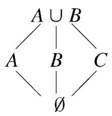
Proof. (a) $\Rightarrow$ (b) is trivial.

(c) $\Rightarrow$ (a) Consider the map $\alpha : Pow(\mathscr{F}) \rightarrow X$ the ambient space defined by union. Readily α is order preserving. Suppose $A_1 \cup \cdots \cup A_k = B_1 \cup \cdots \cup B_m$, $A_i, B_j \in \mathscr{F}$. We claim $k=m$ and the A_i's and B_j's are identical, given some appropriate permutation. If not, there exist some $A_i \neq B_j$, $1 \leqslant j \leqslant m$. So, $A_i \subseteq A_1 \cup \cdots \cup A_k = B_1 \cup \cdots \cup B_m$, contradicting (c). Hence (c) implies that α, the union map, is 1–1 and thus a Boolean lattice isomorphism.

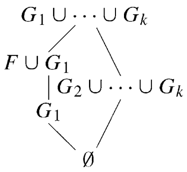
(b) $\Rightarrow$ (c) We prove the contrapositive. Assume some member $F \in \mathscr{F}$ is covered by other members, e.g. $F \subseteq G_1 \cup G_2 \cup \cdots \cup G_j$. We show that $\mathscr{F}.U$ either contains an M_5 or an N_5, and hence must be non-distributive.

We must first dispose of:

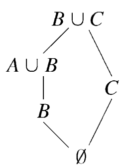
Case 1: There are three sets $A, B, C \in \mathscr{F}$ with identical pairwise unions, i.e. $A \cup B = A \cup C = B \cup C$. Since $\mathscr{F}$ is an anti-chain, A, B, C cover the empty union $\emptyset \in \mathscr{F}.U$. Hence $\mathscr{F}.U$ contains

M_3: 

Case 2: We can assume no 3 sets have identical pairwise unions. Choose $F \subseteq G_1 \cup \cdots \cup G_k$, where k is as small as possible. First suppose $k > 2$. Since $k > 2$, $F \cup G_1$ can contain no member of $\mathscr{F}$, else k would not be minimal. Thus $(F \cup G_1) \cap (G_2 \cup \cdots \cup G_k)$ in $\mathscr{F}.U$ must be empty. Further $G_1 \not\subseteq G_2 \cup \cdots \cup G_k$ by minimality of k. Hence, we have

N_5: 

Finally, let $k = 2$, so $A = B \cup C$. Since we are excluding case 1, either $B \not\subseteq A \cup C$ or $C \not\subseteq A \cup B$. Wlog assume the latter, so we have

N_5: 

□

This theorem need not be true if pseudo-convexity is the closure operator, as Fig. 2 illustrates. The closed subset $\{cfgh\}$ is contained in the union $\{bcfh\} \cup \{efgh\}$. Readily, the lattice of closed subsets, $\mathscr{L}_{\mathscr{C}}$, is not meet distributive.

4. Concept lattice closure

In this section, we examine an important class of closure spaces that are not uniquely generated. Let R be a binary relation between any two sets X and Y, as in Fig. 3. One can form the Galois closure, φ_R, of X with respect to R by generating all the closed sets $Z \subseteq X$ of the form $Z = X_i.\overline{R}.\overline{R^{-1}}$, for $X_i \subseteq X$, where $X_i.\overline{R} = \bigcap_{x \in X_i} x.R \subseteq Y$ and $Y_i.\overline{R^{-1}} = \bigcap_{y \in Y_i} y.R^{-1} \subseteq X$. Alternatively one can form the closure, $\varphi_{R^{-1}}$ of Y with respect to R consisting of the closed sets $Z' = Y_k.\overline{R^{-1}}.\overline{R}$. In formal concept analysis [6], it is customary to regard X as a set of *objects* and Y as a set of *attributes*. Then, the set $X.\overline{R}$ denotes the set of all attributes shared by every object in X. Consequently, $X.\varphi = X.\overline{R}.\overline{R^{-1}}$ denotes the set of *all* the objects that share (at least) these common attributes. Conversely, $Y.\overline{R^{-1}}$denotes the set of all objects sharing every attribute in Y and $Y.\varphi = Y.\overline{R^{-1}}.\overline{R}$ consists of *all* the attributes shared by the objects which (at least) have Y in common.

Ganter and Wille [6] show that φ_R and $\varphi_{R^{-1}}$ are indeed closure operators, and constitute a Galois connection. These closure systems are isomorphic and can be represented by the following lattice of closed sets, partially ordered by inclusion.[4] Labeling each node is the pair of closed sets that is joined by the Galois connection, for example $\langle abg, 123\rangle$. In this case we have oriented the lattice with respect to Y, the set of attributes, with the universe $Y = abcdefghi$ (which must be closed) as the lattice *supremum* and the singleton set $\{a\}$ as the lattice *infimum*. It is partially ordered with respect to set inclusion.

There are no meet distributive sublattices in the lattice of Fig. 4; it is not hard to verify that condition (c) of Theorem 3 is never satisfied. It is apparent that the faces of *abgh* are b and h; while the faces of *abcdf* are b, c, and *df*. Consequently by Theorem 2 these constitute the family of minimal blockers of the generators of these closed sets.

Theorem 4. *Let Z be a closed set with respect to φ. A set $G \subseteq Z$ is a generator of Z if and only if G is a blocker of $\mathscr{F} = \{\Delta_k\}$, the family of faces of Z.*

And, G is a minimal generator if and only if it is a minimal blocker.

	a	b	c	d	e	f	g	h	i
1	×	×					×		
2	×	×					×	×	
3	×	×	×				×	×	
4	×		×				×	×	×
5	×	×		×		×			
6	×	×	×	×		×			
7	×		×	×	×				
8	×		×	×		×			

Fig. 3. A small binary relation R from $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ to $Y = \{a, b, c, d, e, f, g, h, i\}$.

[4] This complete example has been taken from Ganter and Wille's book *Formal Concept Analysis* [6].

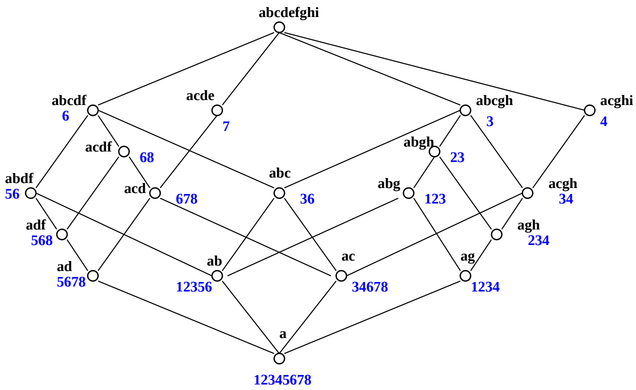


Fig. 4. Closure lattice arising from the Galois closure of the relation $R(X, Y)$.

Proof. Let $G \subseteq Z$ be any generator, i.e. $G.\varphi = Z$. We claim that G is a blocker of $\mathscr{F}$. Suppose not, that for some k, $G \cap \Delta_k = \emptyset$. Then $G \subseteq Z - \Delta_k = Y$. But, since Δ_k is a face, $Y = Z - \Delta_k$ is closed, and $G \subseteq Y$ implies $G.\varphi \subseteq Y \subset Z$, contradicting the premise that G is a generator.

Conversely, let $G \cap \Delta_k \neq \emptyset$ for all $\Delta_k \in \mathscr{F}$. $G \subseteq Z$ implies $G.\varphi \subseteq Z$. Suppose the containment is proper. Then, $G.\varphi = Y \subset Z$, where Y is a maximal contained closed subset. $Z - Y = \Delta_k$ for some k. $G \subseteq Y$ implies $G \cap \Delta_k = \emptyset$, contradicting the premise that G is a blocker.

The last assertion is trivially obvious. □

Consequently, we can see that the set *bh* is the unique generator of *abgh*. Because b, c and *df* are faces of *abcdf*, as can be seen from inspection of Fig. 4, its family of non-unique generators is $\{abcdf\}.\Gamma = \{bcd, bcf\}$.

Readily, the sets *acde* and *acghi* are uniquely generated by *e* and *i*, respectively. Examination of the relation *R* in Fig. 3 shows that any object with attribute *e* must have attributes *acd* as well (there is only one, object 7); and similarly the single object, 4, with attribute *i* also has attributes *acgh*. We can similarly apply Theorem 4 to determine some of the non-unique generators. For example,

Closed set Z	Faces Δ_k	Generating sets $Z.\Gamma = \{Z.\gamma_k\}$
abdf	b, df	bd, bf
abgh	b, h	*bh*
abcdf	b, c, df	bcd, bcf
abcgh	b, c, gh	bcg, bch
acghi	*i*	*i*
abcdefghi	$bdef, defi, eghi, bfghi$	$be, bi, dg, dh, di, ef, eg, eh, ei, fg, fh, fi$

That is, *bd* or *bf* implies *abdf*; *i* implies *aeghi*; and any of $be, \cdots fi$ imply *Y*.

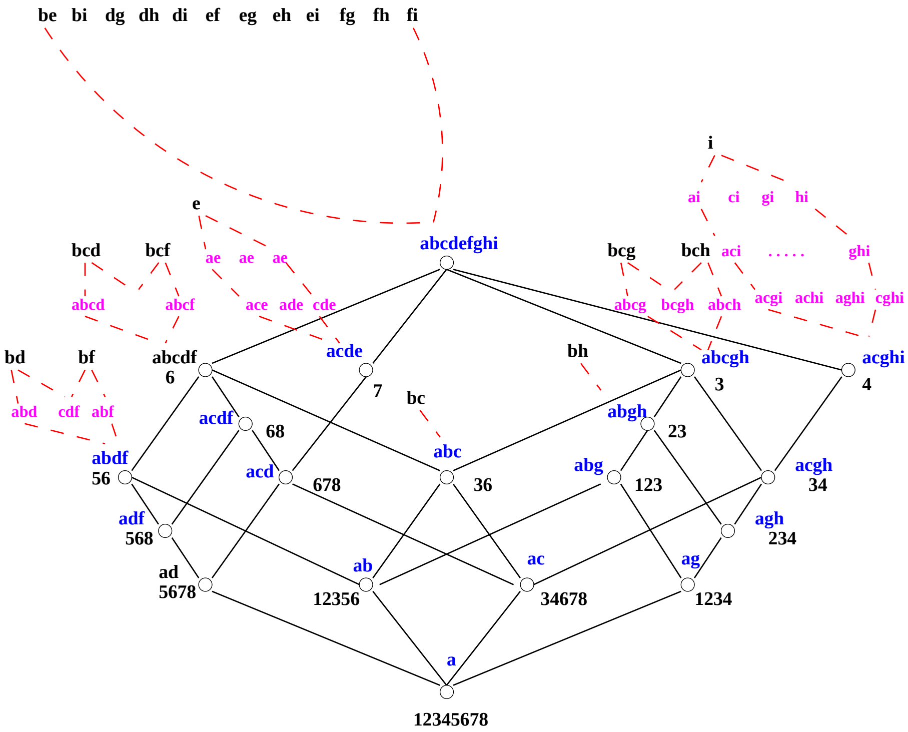


Fig. 5. Formal concept lattice (Fig. 4) with some generators indicated.

Sometimes, as in Figs. 1 and 2, we indicate the generating sets for the closed sets of the lattice. It graphically portrays the implied inference relationships. In Fig. 5 we only indicate a few of these generators by dashed structures growing diagonally to the upper left.

Figs. 2 and 5 have a superficial semblance; but φ_R and φ_η are very different closure operators. This is most easily seen if we regard the binary relation, R, as a bipartite graph on $N = X \cup Y$. Then $X.R = X.\eta$ and $Y.R^{-1} = Y.\eta$. Thus we see that

$$x' \in X.\varphi_\eta \quad \text{if } x'.\bar{\eta} \subseteq \bigcup_{x \in X} x.\bar{\eta},$$

while

$$x' \in X.\varphi_R \quad \text{if } x'.\eta \supseteq \bigcap_{x \in X} x.\eta.$$

Neighborhood domination closure, φ_η, on bipartite graphs such as this one is seldom interesting; and relational closure, φ_R, on general networks is seldom informative either.

We are much more accustomed to having implications associated with attributes and propositions than with objects. Consequently, the preceding discussion becomes more interesting if we regard X as a set of specific objects $\{o_i\}$ and we associate propositions about those objects with the elements of Y. Ganter and Wille [6] actually derived the relation R

from a set of objects discussed in an educational film "Living Beings and Water" and the properties attributed to them in that film. Here the 8 objects of X were (1) *leech*, (2) *bream*, (3) *frog*, (4) *dog*, (5) *spike-weed*, (6) *reed*, (7) *bean* and (8) *maize*, respectively. Their properties, denoted by a through i, were (a) *needs water to live*, (b) *lives in water*, (c) *lives on land*, (d) *needs chlorophyl to prepare food*, (e) *two little leaves grow on germinating*, (f) *one little leaf grows on germinating*, (g) *can move about*, (h) *has limbs* and (i) *suckles its offspring*.

Now, some of the implications indicated above make intuitive sense. In this world, all objects o "need water to live" (a), so $P_a(o)$ is a tautology, where $P_a(o)$ is interpreted as the predicate "object o has attribute/property a". If an object "lives in water (b) and germinates with one leaf (f), then $P_b(o) \wedge P_f(o) \rightarrow P_d(o) \equiv$ "needs chlorophyl", and so on. Consequently, Theorem 4 introduces implication closure [17] into the formal concept structure in a very natural way.

If the closure system is antimatroid, all implications are of the form $P_1 \wedge P_2 \wedge \cdots P_k \rightarrow Q$ where the precedent is a Horn clause. When the closure system is not uniquely generated, disjunctive precedents, such as $(P_b \wedge P_d) \vee (P_b \wedge P_f) \rightarrow P_a \wedge P_b \wedge P_d \wedge P_f$, are possible. The advantages of inference systems based only on Horn clauses are well known [5]. They make implicit use of the antimatroid properties of the implication closure space [17].

In this universe, no object exhibits all the attributes $abcdefghi$, so it represents a logical contradiction *over this universe*. So too, the 12 generating pairs $P_b(o) \wedge P_e(o), \ldots, P_f(o) \wedge P_i(o)$, must each be logical contradictions because they generate *abcdefghi* and because no object these pairs of attributes. This can be also intuitively determined by inspection of the attributes themselves.

Application of Theorem 4 has introduced an easy way of associating a formal concept lattice with a system of implications in which $Y.\Gamma$ denotes the (possibly disjunctive set of) premises and $Y.\varphi$ denotes the transitive closure of those premises. Because these logical implications are valid only in the specific context denoted by the lattice, they may be far richer, more varied, and informative than a logic based on universal satisfiability.

Antimatroid closure spaces, or convex geometries, are important mathematical systems with delightful properties. But, even closure spaces that are not uniquely generated can be useful as well.

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