Closure Lattices*

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Abstract

Closure spaces have been previously investigated by Paul Edelman and Robert Jamison as "convex geometries". Consequently, a number of the results given here duplicate theirs. However, we employ a slightly different, but equivalent, defining axiom which gives a new flavor to our presentation.

The major contribution is the definition of a partial order on *all* subsets, not just closed (or convex) subsets. It is shown that the subsets of a closure space, so ordered, form a lattice with regular, though non-modular, properties. Investigation of this lattice becomes our primary focus.

1 Introduction

We let $\mathbf{U}$ denote some **universe** of interest, that is a set of elements, points, or phenomena. Individual points of $\mathbf{U}$ will be denoted by lower case letters: $a, b, ..., p, q, ... \in \mathbf{U}$. Elements of the power set, $2^{\mathbf{U}}$, we will denote by upper case letters: $..., X, Y, Z \subseteq \mathbf{U}$ (or $\in 2^{\mathbf{U}}$).

Our goal will be to partially order these power set elements. A straightforward partial order by inclusion yields a relatively uninteresting boolean lattice, B_n. If, instead, one looks at some underlying structure of the points in $\mathbf{U}$, then uses this to determine the partial order, more interesting results can be obtained.

In [18] the author defined a convexity concept in directed graphs and demonstrated that the collection of convex subsets, partially ordered by inclusion, formed a lower semi-modular lattice. Edelman [6] independently demonstrated the more general result that any lattice of closed sets would be lower semi-modular if the closure operator satisfied an anti-exchange property. He and Jamison refined these ideas to develop a theory of convex geometries [7].

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The relationship between convex geometries, anti-matroids and matroids is well covered by Korte, Lovasz and Schrader in [15].

In all these cited works, the partial order on the power set is by subset inclusion and it is only the lattice of closed subsets that has interesting structure. Adachi, in [1], proposed a different partial order which explicitly involved a closure operator.[1] But, the power set so ordered is only a semi-lattice. In this paper we introduce a partial ordering of the subsets of $2^{\mathbf{U}}$ induced by any closure operator φ. It will be shown that if φ is "uniquely generated" then $2^{\mathbf{U}}$, so ordered, is a lattice $\mathcal{L}$ whose sublattice of closed elements is precisely that of Edelman. Moreover it will be shown that the structure of $\mathcal{L}$ over non-closed elements has a regularity that permits the enumeration and reconstruction of uniquely generated closures on n elements.

2 Closure Operators

By a **closure** operator, φ, we mean an operator $2^{\mathbf{U}} \xrightarrow{\varphi} 2^{\mathbf{U}}$ satisfying the standard closure axioms:

C1: $X \subseteq X.\varphi$
C2: $X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$
C3: $X.\varphi.\varphi = X.\varphi^2 = X.\varphi$

which are commonly called the Kuratowski Closure Axioms.[2] A set $X \subseteq \mathbf{U}$ is said to be **closed** if $X.\varphi = X$. The pair $(\mathbf{U}, \varphi)$ is called a **closure space** [10].

Closure operators are common in mathematics and other disciplines. For example, the spanning operator of linear algebra is a closure operator, as are reachability operators in graph theory, and all convex hull operators. In computer science, the transaction operator of concurrent processing is a closure operator as are certain greedy algorithms.

The following lemma reviews a number of closure properties that are virtually immediate from the axioms C1, C2, and C3.

Lemma 2.1 *The following are basic closure properties:*

(a) If C is closed and $X \subseteq C \subseteq X.\varphi$ then $C = X.\varphi$.
(i.e. $X.\varphi$ is the smallest closed set containing X.)

[1] Adachi developed his paper with respect to only a single "lower ideal" closure operator, but it can be easily extended to any uniquely generated closure operator.

[2] Note that we are using the standard algebraic notation found in [9] [12], in which binary operations are denoted by infix expressions and unary operations are denoted by suffix expressions. This simplifies notation when closure is composed with other operators. The, technically redundant, dot delimiter facilitates automatic parsing in the kind of computer applications for which this theory is being developed.

(b) $X.\varphi \cup Y.\varphi \subseteq (X \cup Y).\varphi$.
(c) $(X \cap Y).\varphi \subseteq X.\varphi \cap Y.\varphi$.
(d) $X.\varphi \cap Y.\varphi$ *is closed.*
(e) $X.\varphi \cap Y.\varphi = \emptyset$ *for any* X, Y *implies* $\emptyset.\varphi = \emptyset$.
(f) $\mathbf{U}.\varphi = \mathbf{U}$.

It is well known (c.f. [7] [15]) that a family $\mathcal{F}$ of *closed* sets satisfying (d) and (f) with a closure operator defined $X.\varphi = \bigcap_{Y \in \mathcal{F}} \; X \subseteq Y$ is an equivalent axiomatization of closure. Several authors also choose to make $\emptyset.\varphi = \emptyset$ an axiom.

Given an arbitrary closure operator, φ, we define an ordering, $\leq_\varphi$, of $2^{\mathbf{U}}$ by

$$X \leq_\varphi Y \qquad \text{if and only if} \qquad Y \cap X.\varphi \subseteq X \subseteq Y.\varphi \tag{1}$$

This somewhat unusual definition is central to our development. For example, Adachi's definition was $X \leq Y$ if $Y \cap X.\varphi \subseteq X \subseteq Y$ (omitting the second closure); thereby generating a semi-lattice structure. One can easily show that:

Theorem 2.2 $\leq_\varphi$ *is a partial order relation on* $2^{\mathbf{U}}$.

Proof: Reflexivity and anti symmetry are virtual corollaries of the definition $X \leq_\varphi Y \equiv Y \cap X.\varphi \subseteq X \subseteq Y.\varphi$ and the closure axioms C1-C3.

Transitivity is derived from $X \leq_\varphi Y$ and $Y \leq_\varphi Z$ by manipulating the four equivalent containments to yield $Z \cap X.\varphi \subseteq Y \cap X.\varphi \subseteq X$. The idempotency of φ plus $X \subseteq Y.\varphi$ and $Y \subseteq Z.\varphi$ yields $X \subseteq Z.\varphi$. □

Because the ordering on $2^{\mathbf{U}}$ is not simple subset inclusion we must be careful with ordering relationships; we cannot, for example, assume that $X \cap Y \leq_\varphi X$. The following lemma relates properties of X, Y, and Z as sets to their relative order with respect to $\leq_\varphi$.

Lemma 2.3 *Let* $X, Y, Z \subseteq \mathbf{U}$ *and let* φ *be a closure operator on* $\mathbf{U}$.

(a) $X \subseteq Y \subseteq Z$ *and* $X \leq_\varphi Z$, *imply* $X \leq_\varphi Y$
(b) $X \subseteq Y \subseteq Z$ *and* $Z \leq_\varphi X$, *imply* $Z \leq_\varphi Y$ *and* $Y \leq_\varphi X$
(c) $X \leq_\varphi Y$ *and* $X \leq_\varphi Z$ *imply* $X \leq_\varphi Y \cup Z$
(d) $X \leq_\varphi Z$ *and* $Y \leq_\varphi Z$ *imply* $X \cap Y \leq_\varphi Z$
(e) $X \leq_\varphi Y$ *implies* $X \leq_\varphi X \cup Y \leq_\varphi Y$
(f) $X \leq_\varphi Y$ *implies* $X \cap Y \leq_\varphi Y$
(g) $X \leq_\varphi Y \leq_\varphi Z$ *implies* $X \cap Z \subseteq Y$.

All of the preceding results are based solely on the closure axioms C1, C2, C3, and the definition of $\leq_\varphi$ in (1). To continue, we must restrict our closure operators somewhat. In addition to the three required closure axioms, we might consider any, or all, of the following properties.

C4: if $p, q \notin X.\varphi$ then $q \in (X \cup \{p\}).\varphi$ implies $p \in (X \cup \{q\}).\varphi$
C5: $X.\varphi = Y.\varphi$ implies $(X \cap Y).\varphi = X.\varphi = Y.\varphi$
C6: $(X \cap Y).\varphi = X.\varphi \cap Y.\varphi$
C7: $(X \cup Y).\varphi = X.\varphi \cup Y.\varphi$

The first of these properties, C4, is called the Steinitz-MacLane *exchange* property. It characterizes the development of linear algebras, projective geometries, and matroids in terms of closure concepts [3] [10]. One can also postulate an *anti-exchange* property, of the form

$$\text{if } p, q \notin X.\varphi \text{ then } p \in (X \cup \{q\}).\varphi \text{ implies } q \notin (X \cup \{p\}).\varphi \tag{2}$$

which characterizes *alignments* [8], *convex geometries* [7], and *anti-matroids* [15]. In the following development, we will make exclusive use of C5, which we choose to call the **unique generation** property. However, as asserted in [11] [7],

Theorem 2.4 *A closure operator is uniquely generated if and only if it satisfies the anti-exchange property (2).*

Proof: (Unique generation implies anti-exchange) Let $p, q \notin X.\varphi$, and let $p \in (X \cup \{q\}).\varphi$. Assume $q \in (X \cup \{p\}).\varphi$. Then $(X \cup \{p\}).\varphi = (X \cup \{q\}).\varphi$, so that by the unique generation property $X.\varphi = (X \cup \{p\}).\varphi$ implying $p \in X.\varphi$, a contradiction.

(Anti-exchange implies unique generation) Let $X.\varphi = Y.\varphi$. Let M_X be a minimal set contained in X such that $M_X.\varphi = X.\varphi$. We claim that $M_X \subseteq Y$.
Let $p \in M_X$. We note that $(M_X - \{p\}).\varphi \subset X.\varphi$. Now, suppose $p \notin Y$. Let $\emptyset \subset M' \subseteq Y$ be a minimal set such that $(M_X - \{p\} \cup M').\varphi = X.\varphi$. Let $q \in M'$ and let $Z = M_X - \{p\} \cup M' - \{q\}$. Then $(M_X - \{p\} \cup M' - \{q\}).\varphi = Z.\varphi \subset X.\varphi$. Now, $p, q \notin Z$, but $p \in (Z \cup \{q\}).\varphi = X.\varphi$ and $q \in (Z \cup \{p\}).\varphi = X.\varphi$ contradicting the anti-exchange axiom. Consequently $M_X \subseteq Y$. Since $M_X \subseteq X \cap Y$ implies $M_X.\varphi = X.\varphi \subseteq (X \cap Y).\varphi$, equality holds. □
Consequently, uniquely generated closures are completely equivalent to those of abstract convex geometries. Nevertheless, approaching this material from a different direction leads to different insights that appear to be of value in both lattice theory and computer applications. An example of the latter is the transaction concept of operating systems [17] which permeates concurrent database theory [2]; it is an explicit closure operator.

The last two properties are relatively strong. We would note that (a) C6 clearly implies C5; that (b) Kuratowski [16] originally included C7 as one of the closure axioms, because all closed sets in a topological space satisfy it; and that (c) if a closure operator, φ, satisfies both C6 and C7 then it must be an "identity" operator.[3]

[3] More accurately, φ must have the form $X.\varphi = X \cup S$, where S is any fixed subset. If $S = \emptyset$, so that $\emptyset.\varphi = \emptyset$, then for all X, $X.\varphi = X$, the "identity" operator.

Z is said to **cover** X, which we denote $X \prec_\varphi Z$, if $X \neq Z$ and for any Y such that $X \leq_\varphi Y \leq_\varphi Z$, either $X = Y$ or $Y = Z$. Covering relationships are fundamental to the definition of both modularity and height functions in lattices.

Theorem 2.5 (Fundamental Covering Theorem) *If $p \notin X$ then*

(a) $X \leq_\varphi X \cup \{p\}$ *if and only if* $p \notin X.\varphi$
(b) $X \cup \{p\} \leq_\varphi X$ *if and only if* $p \in X.\varphi$

where (a) is a cover if and only if $(X \cup \{p\}).\varphi \subseteq X.\varphi \cup \{p\}$ *and*
(b) is always a covering relationship.

Moreover, if φ is uniquely generated then (a) and (b) characterize all covering relations in $(2^{\mathbf{U}}, \leq_\varphi)$.

Proof:

(a) Readily $X \subseteq (X \cup \{p\}).\varphi$; thus $X \leq_\varphi X \cup \{p\}$ iff $(X \cup \{p\}) \cap X.\varphi \subseteq X$ iff $p \notin X.\varphi$.
The issue is to establish the covering relationship. Let $(X \cup \{p\}).\varphi \subseteq X.\varphi \cup \{p\}$ and let Y be such that $X \leq_\varphi Y \leq_\varphi X \cup \{p\}$. By lemma 2.3(g), $X \subseteq Y$. We assume that $X \neq Y$, else we are done. For $q \in Y - X, q \notin X.\varphi$ since $X \leq_\varphi Y$. $Y \subseteq (X \cup \{p\}).\varphi \subseteq X.\varphi \cup \{p\}$ by assumption. Thus, if $q \in Y - X, q \in \{p\}$ that is $q = p$. Hence X is covered by $X \cup \{p\}$ in $\leq_\varphi$.

(b) Readily $X \cap (X \cup \{p\}).\varphi \subseteq X$; thus $X \cup \{p\} \subseteq X.\varphi$ iff $p \in X.\varphi$.

Let Y be such that $X \cup \{p\} \leq_\varphi Y \leq_\varphi X$. Again $X \subseteq Y$. Assume $X \neq Y$. Let $q \in Y - X$. $Y \leq_\varphi X$ implies $Y \subseteq X.\varphi$ so in particular $q \in X.\varphi$. $X \cup \{p\} \leq_\varphi Y$ implies $Y \cap (X \cup \{p\}).\varphi \subseteq X \cup \{p\}$. So $q \in X \cup \{p\}$ Thus $q = p$.

Now assume that φ is uniquely generated, and that Y covers X. By lemma 2.3(e), we know $X \leq_\varphi X \cup Y \leq_\varphi Y$, and thus either $X = X \cup Y$ or $Y = X \cup Y$ by the covering property. Simplifying, either $Y \subset X$ or $X \subset Y$.
In the first case, suppose $\exists p \in X - Y$. Let $Z = Y \cup \{p\}$ so that $Y \subset Z \subseteq X$. Since $X \leq_\varphi Y$, by lemma 2.3(b), $X \leq_\varphi Z \leq_\varphi Y$. Thus by the covering assumption, $Z = X = Y \cup \{p\}$ Consequently, case (b) of the proposition holds immediately.

For the case $X \subset Y$, assume that $|Y - X| \geq 2$. Our goal is to show that Y can not cover X.
First, suppose that for some $p, q \in Y - X$, $(X \cup \{p\}).\varphi = (X \cup \{q\}).\varphi$. By unique generation property, $(X \cup \{p\}).\varphi = (X \cup \{q\}).\varphi = X.\varphi$. Let $Z = X \cup \{p\}$. $X \subseteq X \cup \{p\} \subseteq Y$. By lemma 2.3(a), $X \leq_\varphi X \cup \{p\}$, and $X \cup \{p\} \leq_\varphi Y$, since $Y \cap (X \cup \{p\}).\varphi = Y \cap X.\varphi \subseteq X \subseteq X \cup \{p\} \subseteq Y.\varphi$. Thus contradicting the covering assumption.
On the other hand, if we suppose that $(X \cup \{p\}).\varphi \neq (X \cup \{q\}).\varphi$ for all p, q, then by the pigeon hole principle, for at least one p, $(X \cup \{p\}).\varphi \subseteq X.\varphi \cup \{p\}$. Now apply case (a) to

$$X \subseteq X \cup \{p\} \subseteq Y$$

to establish $\leq_\varphi$and contradict the initial covering assumption.

Hence, if Y covers X, $|Y| = |X| \pm 1$. □

As direct corollaries of this fundamental theorem, we have:

Corollary 2.6 *Let Z be closed.*

(a) If $Y \leq_\varphi Z$, then Y is closed and $Y \subseteq Z$.
(b) If $Y_1 \prec_\varphi Z$ and $Y_2 \prec_\varphi Z$, then there exists X such that $X \prec_\varphi Y_1$ and $X \prec_\varphi Y_2$.

Proof:

(a) $Y \leq_\varphi Z$ and Z closed imply $Z \cap \leq_\varphi (Y) \subseteq Y \subseteq Z$, so $Y.\varphi \subseteq X$ and $X \subseteq X$.

(b) By (a) above, Y_1 and Y_2 are closed and contained in Z. Let $Y_1 = Z - \{p\}, Y_2 = Z - \{q\}$, where $p \neq q$. By theorem 2.5(a), $p \notin Y_1.\varphi, q \notin Y_2.\varphi$.
Let $X = Y_1 - \{q\} = Y_2 - \{p\} = Z - \{p, q\}$.

(We must still show that $X \leq_\varphi Y_1$ and $X \leq_\varphi Y_2$.) Suppose $q \in X.\varphi$, but since $X \subseteq Y_2$, we have $q \in Y_2.\varphi$ contradicting observation above. So $q \notin X.\varphi$ and similarly $p \notin X.\varphi$. So $X \leq_\varphi Y_1$ and $X \leq_\varphi Y_2$, and since $(X \cup \{q\}).\varphi = Y_1.\varphi \subseteq X.\varphi \cup \{q\}$, these are covers. □

Corollary 2.7 *Let $Y \prec_\varphi Z$.*

(a) If $Z = Y \cup \{p\}$, then both Y and Z are closed.
(b) If $Z = Y - \{p\}$, then Z is not closed.

Although, we will not establish that $(2^{\mathbf{U}}, \leq_\varphi)$ is a lattice until the following section, we would observe that the interval $[\emptyset, \mathbf{U}]$ consisting of those subsets $Y \leq_\varphi \mathbf{U}$ are precisely the closed subsets of $\mathbf{U}$, by corollary 2.6(a). Moreover they constitute a lower semi-modular sublattice as asserted by 2.6(b) and Thm 3.3 in [6].

3 Generators and Lattices

Let Z be any set closed with respect to φ. By a **generator** of Z, denoted $Z.gen_\varphi$, we mean a minimal set Y such that $Y.\varphi = Z$. With a slight abuse of notation, we shall use the expression $Y.gen_\varphi$ with arbitrary Y, with the understanding that if Y is not closed, this means $Y.\varphi.gen_\varphi$. Moreover, we will normally omit the subscript φ.

Readily, if φ satisfies the C5 closure property, then the generators of closed sets are unique. (Because, if Y_1 and Y_2 are distinct minimal sets such that $Y_1.\varphi = Z = Y_2.\varphi$, then $(Y_1 \cap Y_2).\varphi = Z$ contradicting minimality.) This is the reason we call it the *unique generation* property. Clearly we have the equivalent definition

$$Z.gen_\varphi = \bigcap_i \{Y_i \subseteq \mathbf{U} | Y_i.\varphi = Z.\varphi\}. \tag{3}$$

Lemma 3.1 *If φ is uniquely generated, and if $Z \neq \emptyset$ is closed,*

(a) $p \in Z.gen$ if and only if $Z - \{p\}$ is closed, in which case $Z.gen - \{p\} \subseteq (Z - \{p\}).gen$;
(b) $p, q \in Z.gen$ implies there exist closed sets $Y_p, Y_q \subset Z$ such that $p \in Y_p, q \in Y_q$ and $p \notin Y_q, q \notin Y_p$;
(c) if $\emptyset.\varphi = \emptyset$, there exists $p \in Z$ such that $\{p\}$ is closed.

Proof:

(a) Let $p \in Z.gen$. If $(Z - \{p\}).\varphi = Z$, then $((Z - \{p\}) \cap Z.gen).\varphi = Z$ contradicting minimality of $Z.gen$. And if $p \notin Z.gen$, but $Z - \{p\}$ is closed, we have $Z.gen.\varphi \subseteq Z - \{p\} \neq Z.\varphi$, also a contradiction.

Let Y be a generator for $Z - \{p\}$, so $Z - \{p\}$ is the smallest closed set containing Y. $(Y \cup \{p\}).\varphi = Z.\varphi = Z$ is the smallest closed set containing both Y and p. Hence, by unique generation and minimality of $Z.gen$, $Z.gen \subseteq Y \cup \{p\}$, or $Z.gen - \{p\} \subseteq (Z - \{p\}).gen$.

(b) follows directly from (a). Let $Y_p = Z - \{q\}, Y_q = Z - \{p\}$.

(c) follows from (a) using induction on $|Z|$. The condition is necessary to ensure that $Z.gen \neq \emptyset$. □

In light of the preceding lemma, those points $p \in Z.gen$ could be called the *extreme* points of Z, with the set $Z.gen$ itself called the *minimal spanning set* [7] or *basis* [15] of Z. We prefer the term "generator" because it has fewer other associations.

Lemma 3.2 *If φ is uniquely generated, then*

(a) $X \subseteq Y$, implies $X \cap Y.gen \subseteq X.gen$.
(b) $(X \cup Y).gen \subseteq X.gen \cup Y.gen$.
(c) $X.gen \cap Y.gen \subseteq (X \cap Y).gen$.

Proof:

(a) Let $p \in X \cap Y.gen$. By lemma 3.1, $Y.\varphi - \{p\}$ is closed, and $(Y - \{p\}).gen = Y.gen - \{p\}$. Suppose $p \notin X.gen$, then

$$\begin{aligned} X.gen.\varphi \cup (Y.gen - \{p\}).\varphi &= X.\varphi \cup Y.\varphi - \{p\} \\ &= Y.\varphi \subseteq (X.gen \cup (Y - \{p\}).gen).\varphi. \end{aligned}$$

And, $(X.gen \cup (Y - \{p\}).gen).\varphi \subseteq Y.\varphi$. Thus by unique generation property, $((X.gen \cup Y.gen - \{p\}) \cap Y.gen).\varphi = Y.\varphi$, contradicting the minimality of $Y.gen$.

(b) Let $p \in (X \cup Y).gen$ implying $(X \cup Y).\varphi - \{p\}$ is closed, and that $(X \cup Y - \{p\}).gen = (X \cup Y).gen - \{p\}$. We claim that $p \in X \cup Y$, else $(X \cup Y).\varphi - \{p\}$ is the smallest closed set containing X and Y, a contradiction.

Assume $p \in X$, but $p \notin X.gen$. One contradicts the minimality of $(X \cup Y).gen$ with an argument virtually identical to (b) above.

(c) Similar. □

Finally, to characterize those sets of elements Y_i with the same closure, and generator, in terms of the induced order $\leq_\varphi$, we have

Lemma 3.3 *Suppose Y is not closed, and that $\mathcal{B}_Y$ denotes the poset*

$$\{Y_i | Y.\varphi \leq_\varphi Y_i \leq_\varphi Y.\varphi.gen\},$$

with induced order $\leq_\varphi$. Then $\mathcal{B}_Y \cong B_n$ (boolean algebra on n elements), where $n = |Y.\varphi| - |Y.\varphi.gen|$, and $X \leq Y$ in B_n if and only if $Y \subseteq X$.

Proof: By corollary 2.7, no set in $\mathcal{B}_Y$, except $Y.\varphi$, is closed; $Y_i.\varphi = Y.\varphi$; and all covering relationships $Y_i \prec_\varphi Y_k$ are of the form $Y_k = Y_i - \{p\}$. Consequently, $\forall p \in Y_i - Y.\varphi.gen, Y_i \prec_\varphi Y_i - \{q\}$. (If $p \in Y.\varphi.gen$ then $(Y_i - \{p\}).\varphi \neq Y.\varphi$ by the definition of gen.)
Hence, $\mathcal{B}_Y$ consists of all subsets of $Y.\varphi$ containing $Y.\varphi.gen$, ordered by inverse inclusion, $\supseteq$. □
Or equivalently, any **interval** $[Y.\varphi, Y.gen]$ in $(2^{\mathbf{U}}, \leq_\varphi)$ is a boolean algebra.

If we confine our attention to just closed sets Z, it is easy to show that the **height** (cardinality of a maximal irreducible chain) is $|Z|$, as shown in Thm 2.2 [7]. We, however, want a height function for all subsets in $2^{\mathbf{U}}$, for which the above will be a special case.

Theorem 3.4 *Let $ht(Y)$ denote the length n of a maximal irreducible chain*

$$Y_0 \prec_\varphi Y_1 \prec_\varphi Y_2 \prec_\varphi \ldots \prec_\varphi Y_n = Y$$

then $ht(Y) = 2 \cdot |Y.\varphi| - |Y| - |Y_0|$, and ht is a grading of $(2^{\mathbf{U}}, \leq_\varphi)$. In particular, if $Y_0 = \emptyset.\varphi = \emptyset$, then $ht(Y) = 2 \cdot |Y.\varphi| - |Y|$.

Proof: We prove the special case because it is the more important and because the extra machinery needed for the general case tends to obscure the proof structure, even though it is easy to add. We run an induction on $ht(Y)$.

Let $ht(Y) = 0$, implying $Y = \emptyset, Y.\varphi = \emptyset$, and $2 \cdot |\emptyset| - |\emptyset| = 0$.

Let $ht(Y) = 1$, so $\emptyset \prec_\varphi Y$. By theorem 2.5, $Y = \{p\}$ and $(\emptyset \cup \{p\}).\varphi \subseteq \emptyset.\varphi \cup \{p\}$ or $\{p\}.\varphi \subseteq \{p\}$ (establishing that only closed singleton sets cover $\emptyset$) so $|Y.\varphi| = |Y| = 1$, and $ht(Y) = 2 \cdot |Y.\varphi| - |Y| = 1$.

Assume the induction hypothesis is true for $\forall X$ such that $ht(X) < n$, and let $ht(Y) = n$.

1) Y is closed: By the corollary 2.6(a) $\forall X \prec_\varphi Y$, X is closed, $Y = X \cup \{p\}$, and

$$\begin{aligned} ht(Y) &= 2 \cdot |Y.\varphi| - |Y| \\ &= 2 \cdot (|X.\varphi| + 1) - (|X| + 1) \\ &= 2 \cdot |Y.\varphi| - |X| + 1 \\ &= ht(X) + 1. \end{aligned}$$

2) Y is not closed: Observe that by corollary 2.7, we must have $Y = X - \{p\}$ (or $X = Y \cup \{p\}$), $p \in Y.\varphi$, and $|Y| = |X| - 1$.

2a) $X.\varphi = Y.\varphi$: In this case, $X \in \mathcal{B}_Y$ the boolean algebra of lemma 3.3, and

$$\begin{aligned} ht(X) + 1 &= 2 \cdot |X.\varphi| - |X| + 1 \\ &= 2 \cdot |Y.\varphi| - |Y| \\ &= ht(Y). \end{aligned}$$

2b) $X.\varphi \neq Y.\varphi$: Since $X = Y \cup \{p\}$ and $p \in Y.\varphi$, $X.\varphi \neq Y.\varphi$ implies $p \in Y.\varphi.gen$. By lemma 3.1(a), $X.\varphi \cup \{p\}$ is closed, and so $X.\varphi \cup \{p\} = Y.\varphi$ implying $|Y.\varphi| = |X.\varphi| + 1$. Consequently,

$$\begin{aligned} ht(X) + 1 &= 2 \cdot |X.\varphi| - |X| + 1 \\ &= 2 \cdot (|Y.\varphi| - 1) - (|Y| + 1) + 1 \\ &= 2 \cdot |Y.\varphi| - |Y|. \ \square \end{aligned}$$

As noted earlier, Edelman [6], Edelman and Jamison [7], and the author [18] have shown that the closed sets of $\mathbf{U}$, partially ordered by inclusion, form a lattice and have discussed it some detail. We have observed that the partial ordering on $2^{\mathbf{U}}$ developed by Adachi [1] yields only a semi-lattice. A major result of this paper is the demonstration that for any uniquely generated closure, φ, the partial ordering of $2^{\mathbf{U}}$ defined by (1) is in fact a lattice.

Theorem 3.5 *If φ is uniquely generated then $(2^{\mathbf{U}}, \leq_\varphi)$ is a lattice with*

$$inf(X_1, ..., X_n) = [\bigcap_i (X_i.\varphi) \cap (\bigcup_i X_i)] \cup (\bigcap_i (X_i.\varphi)).gen$$

Proof: Let $I = [(\cap_i X_i.\varphi) \cap (\cup_i X_i)] \cup (\cap_i X_i.\varphi).gen$.
We claim $I.\varphi = \cap_i X_i.\varphi$ because: (a) $(\cap_i X_i.\varphi).gen.\varphi = \cap_i X_i.\varphi$, and (b) the latter intersection of closed sets is closed, so that, $(\cap_i X_i.\varphi).\varphi = \cap_i X_i.\varphi$. From (b) we have that

$$\begin{aligned} (\cap_i X_i.\varphi).\varphi \cap (\cup_i X_i)) &\subseteq (\cap_i X_i.\varphi).\varphi \cap (\cup_i X_i).\varphi \\ &= \cap_i X_i.\varphi \cap (\cup_i X_i).\varphi \\ &\subseteq \cap_i X_i.\varphi. \end{aligned}$$

which combined with (a) yields $I.\varphi \subseteq \cap_i X_i.\varphi$. On the other hand, because $(\cap_i X_i.\varphi).gen \subseteq I$ we have $\cap_i X_i.\varphi \subseteq I.\varphi$.

We must show that $\forall k, I \leq_\varphi X_k$ (or $X_k \cap I.\varphi \subseteq I \subseteq X_k.\varphi$). The first containment follows from

$X_k \cap I.\varphi = X_k \cap (\cap_i X_i.\varphi) \subseteq \cup_k X_k \cap (\cap_i X_i.\varphi) \subseteq I$. The second containment is immediate, because $\cap_i X_i.\varphi \cap (\cup_i X_i) \subseteq X_i.\varphi$ and $(\cap_i X_I.\varphi).gen \subseteq \cap_i X_i.\varphi \subseteq X_i.\varphi$.

Suppose that for all $k, Y \leq_\varphi X_k$, then $Y \leq_\varphi I$ because

(a) $X_k \cap Y.\varphi \subseteq Y$ implies $[\cup_k X_k \cap \cap_k X_k.\varphi] \cap Y.\varphi \subseteq Y$.
And, since $Y \subseteq X_k.\varphi, \forall k$ implies $Y.\varphi \subseteq \cap_i X_i.\varphi$, we have by lemma 3.2(a), $Y.\varphi \cap (\cap_i X_i.\varphi).gen \subseteq Y.\varphi.gen \subseteq Y$. So, $I \cap Y.\varphi \subseteq Y$.

(b) $Y \subseteq X_k.\varphi, \forall k$ implies $Y \subseteq \cap_k X_k.\varphi = I.\varphi$.

Having demonstrated that the inf operator exists, we need only establish the existence of a maximal element. We claim it is $\mathbf{U}.gen$. Let $X \subseteq \mathbf{U}$. Again by lemma 3.2(a), $X.\varphi \cap \mathbf{U}.gen \subseteq (X.\varphi).gen \subseteq X$. And, readily $X \subset \mathbf{U}.gen.\varphi = \mathbf{U}$, so $X \leq_\varphi \mathbf{U}.gen$. □
Note that the dominant term of this *inf* operator, commonly denoted by $\wedge$, is $\cap_i X_i.\varphi$, as one might expect. In the next section, we will give examples which require its restriction to $\cup_i X_i$ to keep the *inf* within the original set, and require its augmentation to include the generators, $(\cap_i X_i.\varphi).gen$. Finally, we observe that $X \cap Y \subseteq X \wedge Y$.

The lattice $(2^{\mathbf{U}}, \leq_\varphi)$ (as described in theorem 3.5) we call the *closure lattice induced by* φ, or more simply the **closure lattice**.

4 Examples

In this section, we examine two representative closure lattices. First, let $\mathbf{U} = \{a, b, c, d\}$. The 16 subsets of $2^{\mathbf{U}}$ and their closures have been listed in the following table.

X	∅	{a}	{b}	{c}
X.φ	∅	{a}	{ab}	{ac}
X	{d}	{ab}	{ac}	{ad}
X.φ	{abcd}	{ab}	{ac}	{abcd}
X	{bc}	{bd}	{cd}	{abc}
X.φ	{abc}	{abcd}	{abcd}	{abc}
X	{abd}	{acd}	{bcd}	{abcd}
X.φ	{abcd}	{abcd}	{abcd}	{abcd}

Table 1: A closure, φ, on $\mathbf{U} = \{a, b, c, d\}$

The reader can verify that φ so defined on this small set really is a closure operator, and that it is uniquely generated. The resulting closure lattice, $\mathcal{L}$, is shown in Figure 1. This figure illustrates several of the results of the preceding sections. The interval $[\emptyset, \{abcd\}]$ consists precisely of the closed subsets of $\mathbf{U}$, and is lower semi-modular as required by corollary 2.6. (In this case it is actually distributive.) This sublattice has been drawn with

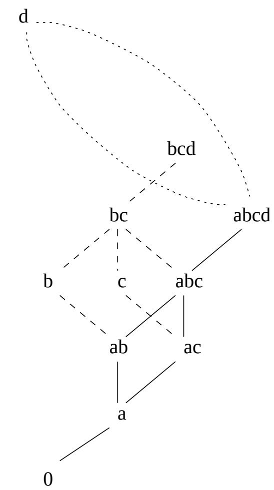

Figure 1: Closure Lattice, given closure φ above

solid lines for emphasis. However, the entire lattice is not lower semi-modular, because $\{bc\}$ covers both $\{b\}$ and $\{c\}$, but neither covers $\{b\} \wedge \{c\} = \{a\}$. Nor is it upper semi-modular.

The subsets $\{b\}, \{c\}, \{bc\}$, and $\{d\}$ are generators for the closed sets $\{ab\}, \{ac\}, \{abc\}$, and $\{abcd\}$ respectively; while $\{a\}$ is its own generator. Except for the element $\{bcd\}$, the boolean algebra comprising the interval $[\{abcd\}, \{d\}]$ (lemma 3.3) has been only schematically indicated as an ellipse to avoid useless clutter.

Observe, that in $\{b\} \wedge \{c\}$, the term $[(\{b\}.\varphi \cap \{c\}.\varphi) \cap (\{b\} \cup \{c\}) = \emptyset$. So, in this case, $\{b\} \wedge \{c\} = (\{b\}.\varphi \cap \{c\}.\varphi).gen$, as required in theorem 3.5.

The preceding closure operator was defined *ex cathedra.* More often they are derived from some underlying relationships or properties of $\mathbf{U}$. On any given universe $\mathbf{U}$ of n points there are a wealth of distinct closure operators, as we will show in the next section. If $\mathbf{U}$ is a partially ordered set, then there are at least 3 natural closure operators corresponding to left ideals, right ideals, and convex intervals.[4] For example, if one defines a left ideal closure

$$Y.\varphi = \{x | (\exists y \in Y)[x \leq y]\}$$

on the 7 point graph of Figure 2 one obtains the somewhat more complex closure lattice of Figure 3 which we will use to motivate the results of the following section. Here again, the sublattice of closed sets, or interval $[\emptyset, \mathbf{U}]$, has been indicated by solid lines, while the boolean algebras, $B_n = [X.\varphi, X.gen]$, have been denoted by dotted "ellipses". Only a few of the covering relationships between "adjacent" closure-generator intervals have been shown.

[4]Many would call these "upper" and "lower" ideals, but when the base universe is ordered, the author orients it from left to right for illustrative purposes in order to minimize confusion with the closure lattice order, which is oriented top down.

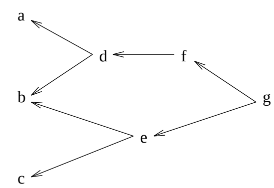

Figure 2: An acyclic graph Γ

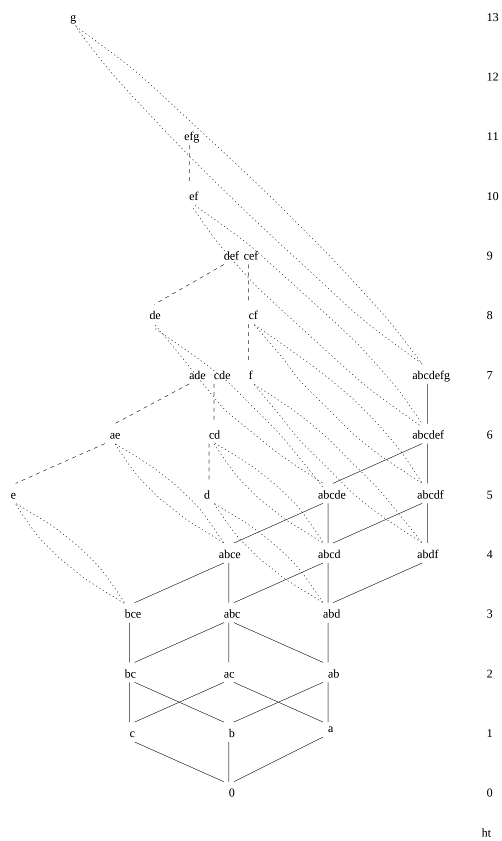

Figure 3: Closure lattice associated with Γ

The compression of a closure lattice with $2^7 = 128$ elements into just 17 closure-generator pairs facilitates an efficient computer representation of closure spaces and their lattices. In light of (1) and (3), a more compressed representation of just the closed sets is also sufficient; but for many applications it is computationally more expensive.

These are not the only closures on 7 points. Clearly the arbitrary distribution of n

points in a Euclidean d-space gives rise to many different convex geometries, e.g. consider the convex sets in 2-space determined by the spatial position of the 7 points in Figure 2. And convex geometries may be generated by undirected graphs with appropriate properties, e.g. block [14] or geodesic graphs, [8].

Processes can also give rise to uniquely generated closures. Both [15] and [5] enumerate various shelling processes that give rise to matroids which satisfy the exchange property, anti-matroids which satisfy the unique generation property, and greedoids (from greedy algorithms) that generalize both.

Some applications have both graph and shelling aspects. For example, one may regard the universe as consisting of the set E of edges of a directed graph, rather than its points or vertices. Then one can recursively define the **transitive closure** of E on P by

$$E.\varphi = \{(x,z) | (\exists y \neq x \in P)[(x,y) \in E, (y,z) \in E.\varphi]\}$$

This corresponds to the customary transitive closure, or *path*, relation. After verifying the three basic closure axioms, one shows

Lemma 4.1 *If $E.\varphi$ is a partial order of P then φ is uniquely generated.*

Proof: Let $E_1.\varphi = E_2.\varphi$ and let them be a partial order on P so that $E|_{P'}.\varphi = E.\varphi|_{P'}$ where $P' \subset P$ is also a partial order.
The lemma is easily verified for any E on small, finite sets P, so inductively assume it is true for all $P' \subset P$.
Readily, $(E_1 \cap E_2).\varphi \subseteq E_1.\varphi$, so we need only show the other containment to establish unique generation. Let $(x,z) \in E_1.\varphi$ implying $\exists y_1 \in P, y_1 \neq x$ such that $(x,y_1) \in E_1, (y_1,z) \in E_1.\varphi$. If $(x,y_1) \in E_2$ we will be done because, first $(x,y_1) \in E_1 \cap E_2$ and, second $E.\varphi$ a partial order implies $E.\varphi|_{P-\{x\}} = E|_{P-\{x\}}.\varphi$, so $(y_1,z) \in E_1.\varphi$ implies $(y_1,z) \in (E_1 \cap E_2).\varphi$. These together imply $(x,z) \in (E_1 \cap E_2).\varphi$.

But (x,y_1) need not be an element of E_2. Then $(x,y_1) \in E_1$ implies $(x,y_1) \in E_2.\varphi$ (since $E_1.\varphi = E_2.\varphi$). So $\exists y_2 \neq x$ and $y_2 \neq y_1$ (since $E_2.\varphi$ is a partial order) such that $(x,y_2) \in E_2, (y_2,y_1) \in E_2.\varphi$. If $(x,y_2) \in E_1 \cap E_2$, we are done, using the argument above. If not, $\exists y_3 \neq x$, $y_3 \neq y_2, y_1$ such that $(x,y_3) \in E_1, (y_3,y_2) \in E_1.\varphi$, etc. In this manner we generate a descending sequence of points

$$x \ldots y_i, y_{i-1}, \ldots y_2, y_1$$

where for even $i, (y_{i+1}, y_i) \in E_1, (y_i, y_{i-1}) \in E_2$. This sequence has a minimal element y, for which $(x,y) \in E_1 \cap E_2$. □

Now, by lemma 3.1(a) one can delete any edge from $E.gen$ to yield a new poset on P with exactly one less edge. This shelling technique has been employed in [4] to generate sequences of posets with n elements.

To see that the condition of Lemma 4.1 cannot be relaxed, consider a directed Peterson graph, as shown in Figure 4 in which E_1 is denoted by solid edges, E_2 is denoted by dashed

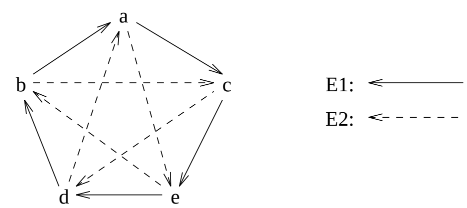

Figure 4: A graph Γ which is not uniquely generated by transitive closure

edges. Readily, $E_1.\varphi = E_2.\varphi$, yet $E_1 \cap E_2 = \emptyset$. Both E_1 and E_2 are minimal generators of this cyclic order.

5 Lattice Structure of Non-Closed Subsets

The structure of the closed sets of any $\mathbf{U}$ with respect to a uniquely generated closure φ is well known. They constitute a lower semi-modular sublattice of $\mathcal{L}$ comprising interval $[\emptyset, \mathbf{U}]$, in which the partial order is subset inclusion, c.f. [7], or corollaries 2.6, 2.7 of the Fundamental Covering Theorem. These can be restated in terms of $X - \{p\}$ rather than $X \cup \{p\}$.

Lemma 5.1 *Let φ be uniquely generated*

(a) $X \prec_\varphi X - \{p\}$ *if and only if* $p \in (X - \{p\}).\varphi$
(b) If X is closed, then
$X - \{p\} \prec_\varphi X$ *if and only if* $p \notin (X - \{p\}).\varphi$
(c) If X is not closed, then
$X - \{p\} \prec_\varphi X$ *if and only if* $(X - \{p\}).\varphi \prec_\varphi X.\varphi$.
(d) If X is closed, then either
$X \prec_\varphi X - \{p\}$ *or* $X - \{p\} \prec_\varphi X$.

We now want to uncover the structural relationships between non-closed elements (sets of $\mathbf{U}$). We know that for any X, the interval $[X.\varphi, X.gen]$ is isomorphic to the boolean algebra B_n (lemma 3.3), but this provides no information regarding the structure between elements in distinct intervals.

Our goal is to show that these [closed set, generator] boolean algebras are stacked, in increasing size, with a covering structure that echoes that of the closed sets which constitute their least elements; that is, the shape of the closure lattices shown in Figures 1 and 3 is not accidental. We begin with

Lemma 5.2 *Let X_1 be closed in $(\mathbf{U}, \varphi)$ and let $Z_1 = X_1.gen$. Let $X_2 \prec_\varphi X_1$ (so $X_2 = X_1 - \{p\}$ is also closed) and let $Z_2 = X_2.gen$. Then for all $Y_2 \in [X_2, Z_2]$, there exists a unique $Y_1 \in [X_1, Z_1]$ such that $Y_2 \prec_\varphi Y_1$.*
Moreover, $Y_1 = Y_2 \cup \{p\}$, and $Y_1 = X_1 - \delta$ where $\delta = X_2 - Y_2$.

Proof: By theorem 2.5, $p \notin X_2.\varphi$. By lemma 3.1(a) $p \in X_1 = X_1.gen$ and $Z_1 - \{p\} \subseteq Z_2$.

Given $Y_2 \in [X_2, Z_2]$, let $Y_1 = Y_2 \cup \{p\}$.
We first claim that $Y_1 \in [X_1, Z_1]$, or equivalently $(Y_2 \cup \{p\}).\varphi = X_1$.
Because $Z_2 \subseteq Y_2 \subseteq X_2$, because $X_2 \cup \{p\} = X_1$, and because $Z_1 - \{p\} \subseteq Z_2$, this follows easily.

We next claim that $Y_2 \prec_\varphi Y_1$. $Y_2 \leq_\varphi Y_1$ because $p \notin Y_2.\varphi = X_2.\varphi$. Moreover, $(X_2 \cup \{p\}).\varphi \subseteq Y_2.\varphi \cup \{p\} = X_2 \cup \{p\} = X_1$. So by theorem 2.5, it is a cover.

Finally, let $Y' \in [X_1, Z_1]$. A corollary to theorem 2.5 is that all covering relationships involve exactly one point, so $Y' = Y_2 \cup \{q\}$. If $q \neq p$ then $Y' \in [X_1, Z_1]$ implies $q \in X_1$, which in turn implies $q \in X_2 = Y_2.\varphi$, thereby contradiction (again by theorem 2.5) that $Y_2 \leq_\varphi Y'$. □

It can be instructive to fill in some of the missing covering relationships of Figure 3 that are asserted by this lemma. For example, the closed element $\{abd\}$ is covered by $\{abcd\}$ and $\{abdf\}$ (with $\delta = \{c\}$ and $\{f\}$ respectively). Consequently, $\{d\}$ is covered by $\{cd\}$ (which is shown) and $\{df\} \in [abdf, f]$ (which is not) respectively. Following are two direct corollaries of this lemma. The first is virtually trivial. The second, which generalizes the structure between elements in different [closed set, generator] intervals is fundamental. We call it the Fundamental Structure Theorem, or FST.

Corollary 5.3 *Let $X \leq_\varphi Y$, then $[X.\varphi, X.gen] \cong B_m$ and $[Y.\varphi, Y.gen] \cong B_n$ where $m \leq n$.*

Theorem 5.4 (Fundamental Structure Theorem) *Let $X.\varphi \leq_\varphi Y.\varphi$ and let $X \in [X.\varphi, X.gen]$. There exists a unique $Y \in [Y.\varphi, Y.gen]$ such that $X \leq_\varphi Y$, where Y is minimal wrt. $\leq_\varphi$ (maximal wrt. $\subseteq$). Moreover $Y = X \cup \Delta$ where $\Delta = Y.\varphi - X.\varphi$ and $Y = Y.\varphi - \delta$ where $\delta = X.\varphi - X$.*

The FST, which is shown by a simple induction argument, asserts the existence of sets *above* any given set in the closure lattice $\mathcal{L}$. For example consider $X = \{ad\} \in [abd, d]$ in Figure 3. Since $\{abd\} \leq_\varphi \{abcde\}$, this theorem asserts that $X \leq_\varphi Y = \{acf\}$ with $\delta = \{b\}$ and $\Delta = \{cf\}$. The unique existence of these elements can be crucial in arguments regarding the continuity of discrete operators, an issue which is not considered in this paper.

Because a closure lattice can be regarded as a nested collection of boolean algebras, that are themselves partially ordered by increasing size, it is possible to explicitly characterize all closure operators on n points. Since each of the $2^{|U|}$ elements of a closure lattice over $\mathbf{U}$ belong to some boolean algebra B_k, we have

Lemma 5.5 *Let φ be any uniquely generated closure operator on* $\mathbf{U}$*, with $n = |\mathbf{U}|$. Let a_k denote the number of [closed set, generator] intervals isomorphic to B_k, then*

$$a_0 \cdot 2^0 \ + \ a_1 \cdot 2^1 \ + \cdots + \ a_{n-1} \cdot 2^{n-1} \ + \ a_n \cdot 2^n = 2^n. \tag{4}$$

The sequence $< a_0, a_1, \cdots, a_n >$ of non-negative integers can be regarded as a *partition* of 2^n. We call it the characteristic **trace** of φ on $\mathbf{U}$. Readily, (a) $a_n \neq 0$ if and only if $a_k = 0$ for all $k < n$, in which case φ is the trivial closure $X.\varphi = \mathbf{U}$ for all $X \subseteq \mathbf{U}$, and $\mathbf{U}.gen = \emptyset$; (b) a_0 denotes the number of closed sets which are their own generators; (c) $a_0 \neq 0$ if and only if $\emptyset.\varphi = \emptyset$; (d) a_0 must be even; and (e) $\sum a_k$ denotes the total number of closed subsets of $\mathbf{U}$ with respect to φ. We observe in passing that (c) and (d) together imply lemma 3.1(c).

One can recursively generate all distinct closure traces, because if $< a_0, \cdots, a_{k-1}, a_k, \cdots, a_n >$ is a characteristic closure trace, then $< a_0, \cdots, 2 \cdot a_{k-1}, a_k - 1, \cdots, a_n >$ is a trace as well. Using a simple program that generates all traces in lexicographic order and counts them, one obtains Table 2. The second column enumerates all closure traces on n points, the

n	all closures	closures with $\emptyset.\varphi = \emptyset$
3	10	6
4	36	26
5	202	166
6	1,828	1,626
7	27,338	25,510
8	692,004	664,666
9	30,251,722	29,559,718

Table 2: Enumeration of distinct characteristic traces of n point closures

third column those with $a_0 \neq 0$, or by the observation above $\emptyset.\varphi = \emptyset$. (For $n > 9$, these values exceed the length of a long integer on the computer used to generate the table.) As mentioned earlier, there exist many different closures on a space of n points.

Given any arbitrary n point closure trace, such as the traces $< 20, 2, 2, 0, 0, 0 >$ or $< 8, 0, 4, 3, 1, 0, 1, 0 >$ (the trace of Figure 3), one can generate actual closure spaces with these characteristics. That is,

Theorem 5.6 *Let $< a_0, a_1, \cdots, a_{n-1}, a_n >$ be any sequence of non-negative integers such that $\sum_{k=0}^{n} a_k \cdot 2^k = 2^n$. There exists a closure operator φ on* $\mathbf{U}$*, $|\mathbf{U}| = n$, for which*

$trace(\varphi) =< a_0, \cdots, a_n >$.

Proof: This is most easily demonstrated by a procedure which actually generates the closure operator φ, or more precisely a collection of [closed set, generator] pairs which defines the operator.

In our implementation, points are lower case letters, $a, b, c, \ldots, z$ and point sets are lexicographically ordered strings of distinct points. Therefore the function *first_points* is well defined; it returns a point set consisting of the first j points according to this arbitrary order.

```
 generate (int n, int a[ ], point_set U)
      Given n and a[0], ... a[n], generate the closure
      pairs of the corresponding closure over U.

point       p
int         i, j, k
set         closed, closure_def
point_set   new_cl, new_gen, old_cl, old_gen
list        queue

closed = empty_set
k = n
if a[k] = 0
     decrement k until a[k] != 0
new_cl = U
new_gen = first_points (j, new_cl )
insert new_cl into closed
add (new_cl, new_gen) to queue
insert (new_cl, new_gen) into closure_def
while queue is not empty do
     {
     remove (old_cl, old_gen) from queue
     for each p in old_gen do
          {
          if a[k] = 0
               decrement k until a[k] != 0
          new_cl = old_cl - {p}
          if new_cl not in closed
               {         (old_cl, old_gen) covers a boolean
                         interval (new_cl, new_gen) which
                         is isomorphic to B[n].
               insert new_cl into closed
               j = size_of(new_cl) - k
               new_gen = first_points (j, new_cl)
               add (new_cl, new_gen) to queue
               insert (new_cl, new_gen) into closure_def
               decrement a[k]
               }
          }
     }
return closure_def
```

To show that the algorithm is correct, one needs only show that for closed X, if $p \in X.gen$ then X covers $X - \{p\}$. But this follows directly from lemma 5.1(b). □

Figure 5 illustrates the closure lattice corresponding to the set of closure pairs $\{[X.\varphi, X.gen]\}$ returned by the *generate* procedure when given the trace $< 20, 2, 2, 0, 0, 0 >$.

Unfortunately, trace sequences satisfying (4) do not uniquely characterize closure lattices. Consider the lattice of Figure 6(b) which also has the trace $< 20, 2, 2, 0, 0, 0 >$ and compare it with Figure 5. Figure 6(b) is obtained from the graph to its left, using a convex

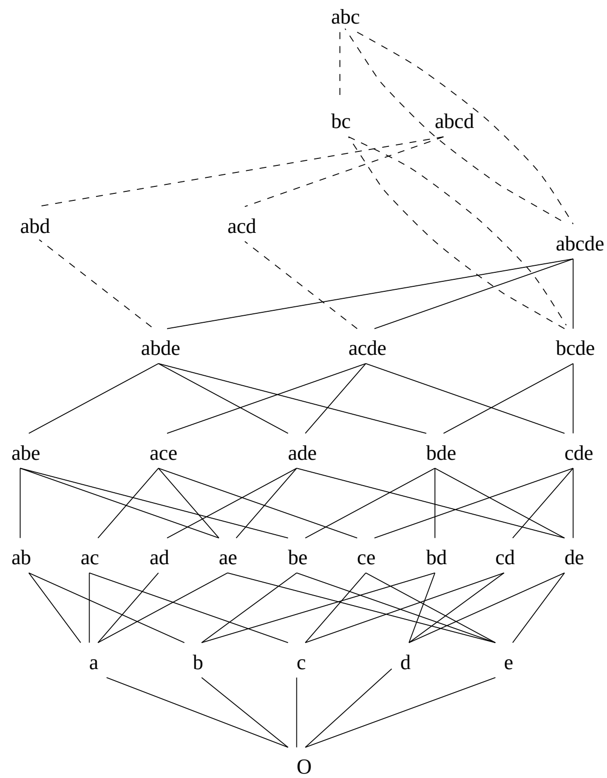

Figure 5: Closure lattice with trace $< 20, 2, 2, 0, 0, 0 >$ returned by the procedure *generate*

interval closure[5]

$$Y.\varphi = \{x | y_1 \leq x \leq y_2, \text{ where } y_1, y_2 \in Y\}.$$

Consequently, Table 2 only can be regarded as providing a lower bound on the number of distinct closure operators, and closure lattices, on n points.

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[5] called *order convex* in [13] and [7].

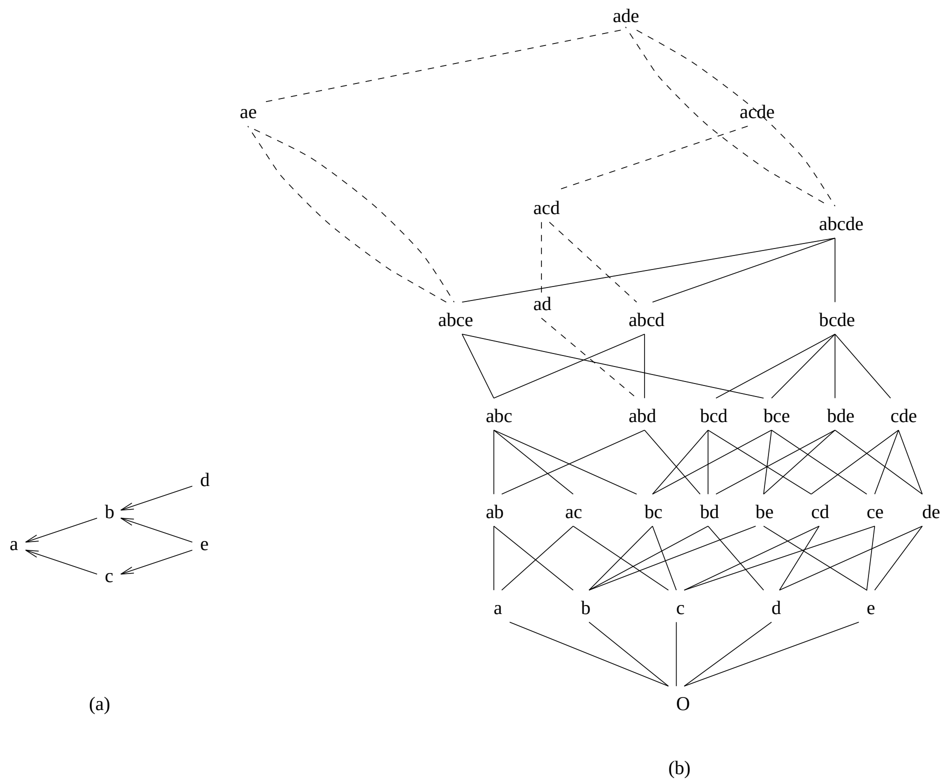

Figure 6: Closure lattice with trace $< 20, 2, 2, 0, 0, 0 >$ induced by convex interval closure on (a)

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