Class 23: Gödel's Theorem

CS150: Computer Science
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Computer Science

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PS5

How are commercial databases different from what you implemented for PS5?

UVa's Integrated Systems Project to convert all University information systems to use an Oracle database was originally budgeted for **$58.2 Million** (starting in 1999). Actual cost is likely to be $100 Million.

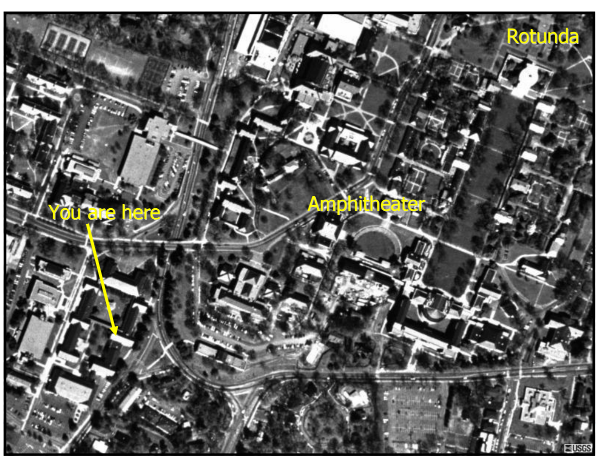
http://www.virginia.edu/isp/

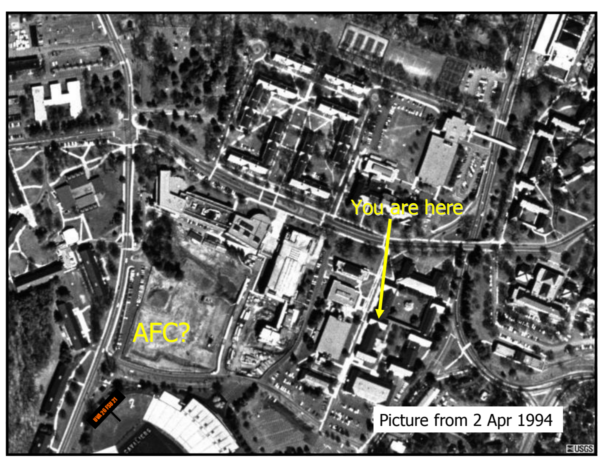
Real Databases

- Atomic Transactions: a transaction may involve many modifications to database tables, but the changes should only happen if the whole transaction happens (e.g., don't charge the credit card unless the order is sent to the shipping dept)
- Security: limit read/write access to tables, entries and fields
- Storage: need to efficiently store data on disk, provide backup mechanisms
- Scale: to support really big data tables, real databases do lots of clever things

How big are big databases?

- Microsoft TerraServer
  - Claimed biggest in 1998
  - Aerial photos of entire US (1 meter resolution)





Big Databases

- Microsoft TerraServer
  - 3.3 Terabytes (claimed biggest in 1998)
  - 1 Terabyte = 2^{40} Bytes ~ 1 Trillion Bytes
- Google Maps
  - Not so big? (but offers programming interface!)
- Wal-Mart
  - 285 Terabytes (2003)
- Stanford Linear Accelerator (BaBar)
  - 500 Terabytes (30 KB per particle collision)

How much work?

- table-select is $\theta(n)$ where n is the number of entries in the table
  - Would your table-select work for Wal-Mart?
  - If 1M entry table takes 1s, how long would it take Wal-Mart to select from 285TB ~ 2 Trillion Entries? 2 000 000s ~ 23 days

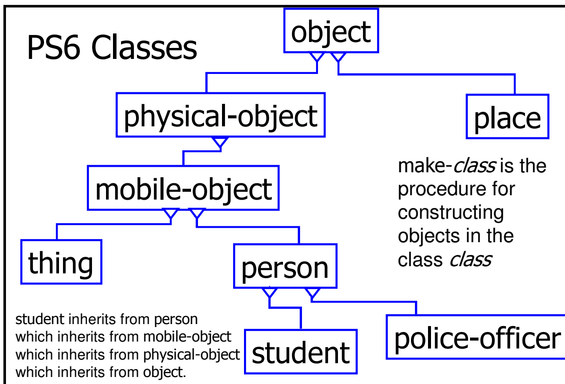
How do expensive databases perform so much faster?

PS6

Make an adventure game programming with objects

Many objects in our game have similar properties and behaviors, so we use inheritance.

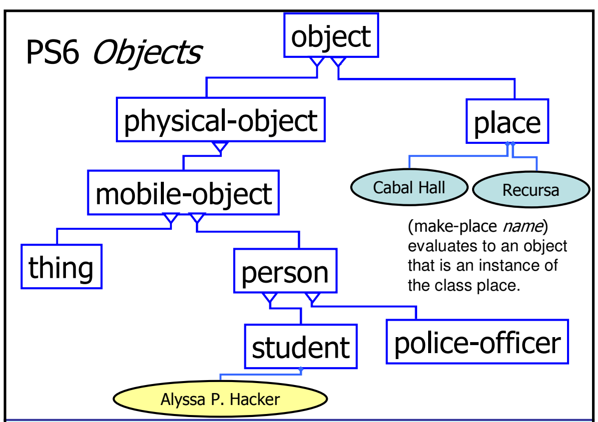
PS6 Classes



make-*class* is the procedure for constructing objects in the class *class*

student inherits from person which inherits from mobile-object which inherits from physical-object which inherits from object.

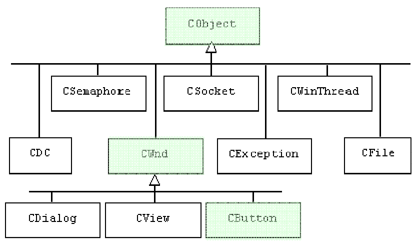
PS6 *Objects*



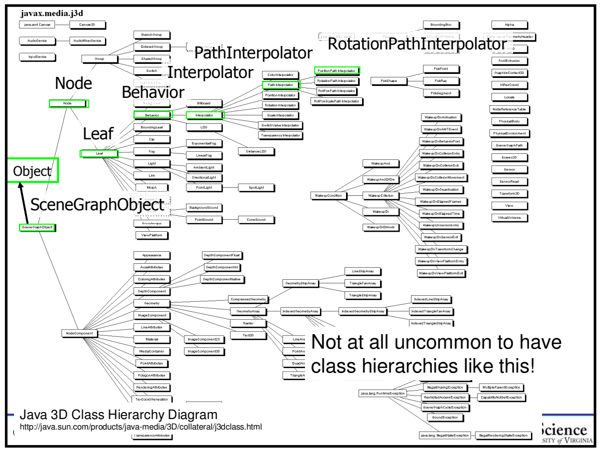
(make-place *name*) evaluates to an object that is an instance of the class place.

Are there class hierarchies like this in the "real world" or just in fictional worlds like Charlottansville?

Microsoft Foundation Classes



CButton inherits from CWnd inherits from CObject
"A button is a kind of window is a kind of object"

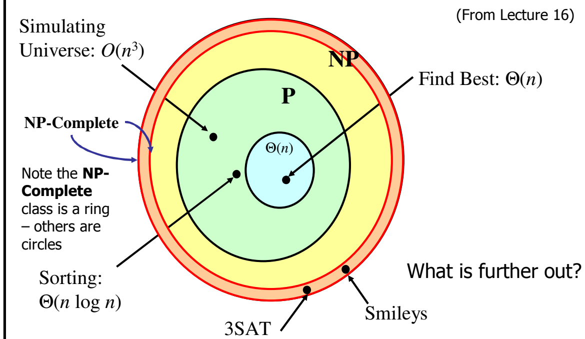


Story So Far

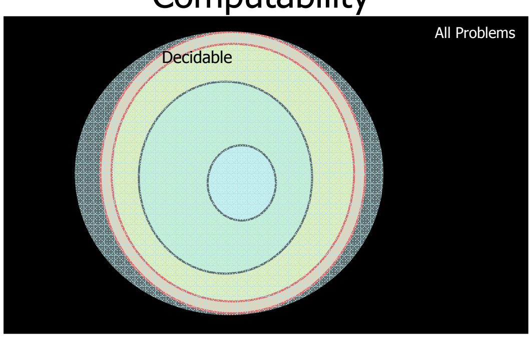
- Much of the course so far:
  - Getting comfortable with recursive definitions
  - Learning to write a program to do (almost) anything (PS1-4)
  - Learning more elegant ways of programming (PS5-6)
- This Week:
  - Getting ***un***comfortable with recursive definitions
  - Understanding why there are some things no one can write a program to do!

Problem Classes if P ≠ NP:

(From Lecture 16)



Computability



Computer Science/Mathematics

- Computer Science (Imperative Knowledge)
  - Are there (well-defined) problems that cannot be solved by *any* procedure?
- Mathematics (Declarative Knowledge)
  - Are there true conjectures that cannot be the shown using *any* proof?

Mechanical Reasoning

- Aristotle (~350BC): *Organon*
  - We can explain logical deduction with rules of inference (syllogisms)

    Every B is A

    C is B

    ➔ C is A

    Every human is mortal.

    Gödel is human.

    Gödel is mortal.

More Mechanical Reasoning

- Euclid (~300BC): *Elements*
  - We can reduce geometry to a few axioms and derive the rest by following rules
- Newton (1687): *Philosophiæ Naturalis Principia Mathematica*
  - We can reduce the motion of objects (including planets) to following axioms (laws) mechanically

Mechanical Reasoning

- Late 1800s – many mathematicians working on codifying "laws of reasoning"
  - George Boole, *Laws of Thought*
  - Augustus De Morgan
  - Whitehead and Russell

All **true** statements about number theory

Perfect Axiomatic System

Derives **all** true statements, and **no** false statements starting from a finite number of axioms and following mechanical inference rules.

Incomplete Axiomatic System

incomplete

Derives **some, but not all** true statements, and **no** false statements starting from a finite number of axioms and following mechanical inference rules.

Inconsistent Axiomatic System

Derives **all** true statements, and **some** false statements starting from a finite number of axioms and following mechanical inference rules.

some false statements

Principia Mathematica

- Whitehead and Russell (1910– 1913)
  - Three Volumes, 2000 pages
- Attempted to axiomatize mathematical reasoning
  - Define mathematical entities (like numbers) using logic
  - Derive mathematical "truths" by following mechanical rules of inference
  - Claimed to be *complete* and *consistent*
    - All true theorems could be derived
    - No falsehoods could be derived

Russell's Paradox

- Some sets are not members of themselves
  - set of all Jeffersonians
- Some sets are members of themselves
  - set of all things that are non-Jeffersonian
- *S* = the set of all sets that are not members of themselves
- Is *S* a member of itself?

Russell's Paradox

- *S* = set of all sets that are not members of themselves
- Is *S* a member of itself?
  - If *S* **is** an element of *S*, then *S* **is** a member of itself and should **not** be in *S*.
  - If *S* **is not** an element of *S*, then *S* **is not** a member of itself, and **should** be in *S*.

Ban Self-Reference?

- *Principia Mathematica* attempted to resolve this paragraph by banning self-reference
- Every set has a type
  - The lowest type of set can contain only "objects", not "sets"
  - The next type of set can contain objects and sets of objects, but not sets of sets

Russell's Resolution?

Set ::= Set_n

Set_0 ::= { x | x is an *Object* }

Set_n ::= { x | x is an *Object* or a Set_{n-1} }

S: Set_n

Is S a member of itself?

No, it is a Set_n so, it can't be a member of a Set_n

Epimenides Paradox

Epidenides (a Cretan):
"All Cretans are liars."

Equivalently:
"This statement is false."

Russell's types can help with the set paradox, but not with this one.

Gödel's Solution

All consistent axiomatic formulations of number theory include *undecidable* propositions.

(GEB, p. 17)

undecidable – cannot be proven either true or false inside the system.

Kurt Gödel

- Born 1906 in Brno (now Czech Republic, then Austria-Hungary)
- 1931: publishes *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme* (*On Formally Undecidable Propositions of Principia Mathematica and Related Systems*)

- 1939: flees Vienna
- Institute for Advanced Study, Princeton
- Died in 1978 – convinced everything was poisoned and refused to eat

Gödel's Theorem

In the Principia Mathematica system, there are statements that cannot be proven either true or false.

Gödel's Theorem

In any interesting rigid system, there are statements that cannot be proven either true or false.

Gödel's Theorem

All logical systems of any complexity are incomplete: there are statements that are *true* that cannot be proven within the system.

Proof – General Idea

- Theorem: In the Principia Mathematica system, there are statements that cannot be proven either true or false.
- Proof: Find such a statement

Gödel's Statement

G: This statement of number theory does not have any proof in the system of *Principia Mathematica*.

G is unprovable, but true!

Gödel's Proof

G: This statement of number theory does not have any proof in the system of *PM*.

If *G* were provable, then PM would be inconsistent.

If *G* is unprovable, then PM would be incomplete.

PM cannot be complete and consistent!

Finishing The Proof

- Turn *G* into a statement in the *Principia Mathematica* system
- Is *PM* powerful enough to express "This statement of number theory does not have any proof in the system of *PM*."?

How to express "does not have any proof in the system of *PM*"

- What does it mean to have a proof of *S* in PM?
  - There is a sequence of steps that follow the inference rules that starts with the initial axioms and ends with *S*
- What does it mean to **not** have **any** proof of *S* in PM?
  - There is **no** sequence of steps that follow the inference rules that starts with the initial axioms and ends with *S*

Can PM express unprovability?

- There is **no** sequence of steps that follow the inference rules that starts with the initial axioms and ends with *S*
- *Yes:* (using Scheme-ified Gödel notation)

```
(unprovable? x) = (not (provable? x))
(provable? x) = (exists (y) (proves y x))
(proves x y) =
  (and (valid-proof-steps x)
       (eq? (apply-proof x initial-axioms) y)))
```


Can we express "This statement of number theory"

- Yes!
  - That's the point of the TNT Chapter in GEB
- We can write turn every statement into a number, so we can turn "This statement of number theory does not have any proof in the system of *PM*" into a number

Gödel's Proof

G: This statement of number theory does not have any proof in the system of *PM*.

If *G* were provable, then PM would be inconsistent.

If *G* is unprovable, then PM would be incomplete.

PM cannot be complete and consistent!

Generalization

All logical systems of any complexity are incomplete:

there are statements that are *true* that cannot be proven within the system.

Practical Implications

- Mathematicians will ***never*** be completely replaced by computers
  - There are mathematical truths that cannot be determined mechanically
  - We can build a computer that will prove only true theorems about number theory, but if it cannot prove something we do not know that that is not a true theorem.

Charge

- Wednesday:
  - Does not being able to prove things mechanically have anything to do with not being able compute things?
  - What is the equivalent to the Gödel sentence for computation?
- Friday:
  - How to prove a problem has no solving procedure