Data-Driven Robust Taxi Dispatch under Demand Uncertainties
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Abstract—In modern taxi networks, large amounts of taxi occupancy status and location data are collected from networked in-vehicle sensors in real-time. They provide knowledge of system models on passenger demand and mobility patterns for efficient taxi dispatch and coordination strategies. Such approaches face a new challenge: how to deal with uncertainties of predicted customer demand while fulfilling the system’s performance requirements, including minimizing taxis’ total idle mileage and maintaining service fairness across the whole city. To address this problem, we develop a data-driven robust taxi dispatch framework to consider spatial-temporally correlated demand uncertainties. The robust optimization problem is concave in the uncertain demand and convex in the decision variables. Uncertainty sets of random demand vectors are constructed from data based on theories in hypothesis testing, and provide a desired probabilistic guarantee level for the performance of robust taxi dispatch solutions. We prove equivalent computationally tractable forms of the robust dispatch problem using the minimax theorem and strong duality. Evaluations on four years of taxi trip data for New York City show that by selecting a probabilistic guarantee level at 75%, the average demand-supply ratio error is reduced by 31.7%, and the average total idle driving distance is reduced by 10.13% or about 20 million miles annually, compared with non-robust dispatch solutions.

I. INTRODUCTION

Modern transportation systems are equipped with various sensing technologies for passenger and vehicle tracking, such as radio-frequency identification (RFID), global positioning system (GPS), and occupancy status sensing systems. Sensing data collected from transportation systems provides us opportunities for understanding spatial-temporal patterns of passenger demand. Methods of predicting taxi-passenger demand [24], [30], travel time [3], [18], [29] and traveling speed [2] according to traffic monitoring data have been developed. Simple travel time predictors were demonstrated to come close to fundamental error bounds in delay prediction [16].

Based on such rich spatial-temporal information about passenger mobility patterns and demand, many control and coordination solutions have been designed for intelligent transportation systems. Coverage control algorithms to allocate groups of vehicles [12], robotic load re-balancing policies that minimize the number of re-balancing trips for mobility-on-demand systems [26], [32], and smart parking systems that allocates resource based on a driver’s cost function [17] have been proposed. Dispatch algorithms that aim to minimize customers’ waiting time [20], [28] or to reduce cruising mile [31] have been developed. Miao et. al design a Receding Horizon Control (RHC) framework of dispatching taxis towards both current and predicted future demand, balancing taxi supply throughout the city and reducing idle cruising distance [23]. Strategies for resource allocation depend on the model of demand in general, and the knowledge and assumptions about the demand affect the performance of the supply-providing approaches [11], [25]. These works heavily rely on precise passenger-demand models to make dispatch decisions.

However, passenger-demand models have their intrinsic model uncertainties that result from many factors, such as weather, passenger working schedule, and city events etc. Algorithms that do not consider these uncertainties can lead to inefficient dispatch services, resulting in long waiting times of under-served passengers, imbalanced workloads, and increased taxi idle mileage. Meanwhile, considering future demand when making the current dispatch decisions helps to reduce resource re-allocation costs [32] and taxis’ total idle distance [23], and a robust allocation scheme shows its advantage in worst-case scenarios compared with non-robust approaches [1]. While robust optimization aims to minimize the worst-case cost under all possible random parameters, it sacrifices average system performances. For a taxi dispatch system, it is essential to address the trade-off between worst-case system performance guarantee and the average dispatch cost under uncertain demand. A promising yet challenging approach is a robust dispatch framework with an uncertain demand model, called an uncertainty set, that captures spatial-temporal correlations of demand uncertainties and provides a probabilistic guarantee (as defined in problem (12)). Solving the robust dispatch problem with the constructed uncertainty set allows us to guarantee that the probability of having an actual dispatch cost under the true demand that smaller than the optimal cost of the robust dispatch solution is greater than a desired value.

In this work, we consider two aspects of a robust taxi dispatch framework given a taxi-operational records dataset: (1) how to formulate a robust resource allocation problem that dispatches vacant taxis towards predicted uncertain demand, and (2) how to construct spatial-temporally correlated...
uncertain demand sets for this robust resource allocation problem without sacrificing too much average performance of the system. We have the freedom to specify a lower bound value for the probability that an actual dispatch cost under the true demand vector being smaller than the optimal cost of the robust dispatch solutions. The data-driven robust dispatch model we design allows us to find a better solution for considering the trade-off between the average dispatch cost and the minimum cost under the worst-case scenario than previous methods that do not provide any guarantees.

We first develop the objective and constraints of a multi-stage robust dispatch problem considering spatial-temporally correlated demand uncertainties. The objective of a system-level optimal dispatch solution is balancing workload of taxis in each region of the entire city with minimum total current and expected future idle cruising distance. We then design a data-driven algorithm for constructing uncertainty demand sets without assumptions about the true distribution of the demand vector. The constructing algorithm is based on theories proved for independent and identically distributed (i.i.d.) sampled random vectors in the robust optimization literature [5], [14], [27]. However, how to apply these theories for spatial-temporal data and a robust resource allocation form of taxi dispatch problem based on the constructed spatial-temporally correlated uncertainty sets have not been explored before. To the best of our knowledge, this is the first work to design a robust taxi dispatch framework that provides a desired probabilistic guarantee using predictable and realistic demand uncertainty sets.

With two types of uncertainty sets — one box type and one second-order-cone (SOC) type, we prove equivalent convex optimization forms of the robust dispatch problem via the minimax theorem and the strong duality theorem. The robust dispatch problem formulated in this work is convex over the decision variables and concave over the constructed uncertain sets with decision variables on the denominators. This form is not the standard form (i.e., linear programming (LP) or semi-definite programming (SDP) problems) that has already been covered by previous work [5], [8], [13]. With proofs shown in this work, both system performance and computational tractability are guaranteed under spatial-temporal demand uncertainties. Based on four years of taxi trip data in New York City, we evaluate factors that affect the accuracy of the uncertain demand sets, properties of each type of uncertainty sets, and trade-off between the probabilistic guarantee levels and the minimum cost under the worst-case scenario than previous methods that do not provide any guarantees.

The rest of the paper is organized as follows. The taxi dispatch problem is described and formulated as a robust optimization problem given a closed and convex uncertainty set in Section II. The requirement of modeling uncertain demand sets are described in Section III followed by the algorithm for constructing uncertain demand sets based on taxi operational records data in Section IV. Equivalent computationally tractable forms of the robust taxi dispatch problem given different forms of uncertainty sets are proved in Section V. Evaluation results based on a real data set are shown in Section VI. Concluding remarks are provided in Section VII.

### Notations

For any vector $x$, we denote by $x^T$ the transpose of $x$, and $x_i$ as the $i$-th component of $x$. We denote $1_n \in \mathbb{R}^n$ as a vector of all ones.

## II. Problem Formulation

The goal of taxi dispatch is to direct vacant taxis towards current and future passengers with minimum total idle mileage. There are two objectives. One is sending more taxis for more requests to reduce mismatch between supply and demand across all regions in the city. The other is to reduce the total idle driving distance for picking up expected passengers in order to save cost. Involving predicted customer demand of the future when making current decisions benefits to increasing total profits, since drivers are able to travel to regions with better chances to pick up future passengers. In this section, we formulate a taxi dispatch problem with uncertainties in the predicted spatial-temporal patterns of passenger demand. A typical monitoring and dispatch infrastructure is shown in Figure 1. The dispatch center periodically collects and stores real-time information such as GPS location, occupancy status and road conditions; dispatch solutions are sent to each taxi via cellular radio. An RHC framework that cooperates predicted demand model and real-time sensing data is designed in [23], where either a deterministic model or an uncertain demand model is applied to calculate a dispatch solution at each step of sliding the time window and updating the latest sensing information. However, the robust dispatch problem formulated in [23] does not provide any probabilistic guarantee as the model we design in this work. We define the problem of finding a robust dispatch in the rest of this section, which is compatible with the RHC framework of [23].
A. Problem description

We discretize time and space in problem formulation for computational efficiency. We assume that the entire city is divided into \( n \) regions, and discrete time slots are indexed by \( k = 1, 2, \ldots, \tau \). Typically, it is difficult to predict a deterministic value of passenger demand of a region during specific time. With prior knowledge and data sets, we assume that the passenger demand model is described by uncertainty vectors belonging to closed and convex sets defined as

\[
\Delta_k \subset \mathbb{R}^n, \quad k = 1, \ldots, \tau,
\]

where \( \Delta_k \) is the number of total requests within region \( j \) during time \( k \), and \( \tau \) is the model predicting time horizon. Here we relax the integer constraint of \( \Delta_k \in \mathbb{N} \) to positive, since constructing an uncertainty set for a continuous vector is more convenient and this relaxation provides an accurate enough demand model. The total number of requests at region \( j \) may have similar patterns as its neighbors, for instance, during busy hours, several regions locate in downtown area may all have peak demand. This type of spatial correlations of demand across each region during the same time slot \( k \) is reflected by the correlation of each element of \( \Delta_k \). Meanwhile, demand can also be temporal correlated, that demand during several consecutive time slots \( \Delta_k, k = 1, \ldots, \tau \) may show similar characteristics like busy hours. Hence, it is possible to describe both spatial and temporal correlations by one set \( \Delta \) for uncertain demand vectors \( \Delta_k, k = 1, \ldots, \tau \). We define the concatenation of sequences \( (r^1, \ldots, r^\tau) \) as

\[
r_c = [(r^1)^T, (r^2)^T, \ldots, (r^\tau)^T]^T \in \Delta \subset \mathbb{R}^{\tau n},
\]

and each closed, convex set \( \Delta_k \) is a projection of \( \Delta \)

\[
\Delta_k := \{ r^k \mid \exists r^1, \ldots, r^{k-1}, r^{k+1}, \ldots, r^\tau, \text{ s.t. } r^k \in \Delta \}.
\]

The closed and convex form of \( \Delta \) depends on the method and theory applied to construct the uncertainty set, which we will describe in Section III.

Considered as one type of resource allocation problem, the basic idea of a robust dispatch model that balances taxis’ supply in a network flow model is described in Figure 2. The dispatch framework decides the amount of vacant taxis that should traverse between each node pair according to the demand at each node according to control requirements and practical constraints. The edge weight of the graph represents the distance between two regions. Specifically, each region has an initial number of vacant taxis provided by real-time sensing information and an uncertain predicted demand.

We define a non-negative decision variable matrix \( X^k \in \mathbb{R}^{n \times N} \), \( X^k_{ij} \geq 0 \), where \( X^k_{ij} \) is the number of taxis (amount of resource) dispatched from region \( i \) to region \( j \). We relax the integer constraint of \( X^k_{ij} \in \mathbb{N} \) to a non-negative constraint, since mixed integer programming is not computational efficient for a large-scale robust optimization problem. In this work we consider the following robust resource allocation problem

\[
\min \ \max \ \min \ \max \ \ldots \ \min \ \max \ \ \ X^k, \quad r^1 \in \Delta_1, \quad X^2, \quad r^2 \in \Delta_2, \quad \ldots, \quad X^\tau, \quad r^\tau \in \Delta_c \sim \mathcal{D}_c \sim \mathcal{E}_c.
\]

\[
J = \sum_{k=1}^{\tau} (J_D(X^k) + \beta J_E(X^k, r^k)) \quad (1)
\]

s.t. \( X^k \in \mathcal{D}_c \).

where \( J_D \) is a convex cost function for allocating or re-allocating resources, \( J_E \) is a function concave in \( r^k \) and convex in \( X^k \) that measures the service fairness of the resource allocating strategy, and \( \mathcal{D}_c \) is a convex domain of the decision variables that describes the constraints of the resource allocating strategies. We define specific formulations of the objective and constraint functions for a robust taxi dispatch problem in the rest of this section.

B. Robust taxi dispatch problem formulation

**Estimated cross-region idle-driving distance:** When traversing from region \( i \) to region \( j \), taxi drivers take the cost
of cruising on the road without picking up a passenger till the target region. Hence, we consider to minimize this kind of idle driving distance while dispatching taxis. We define the weight of cruising on the road without picking up a passenger till the region \(i\) at time \(k\) is denoted by \(W_{ij}\).

The distance every taxi can drive should be bounded by a threshold parameter \(m \in \mathbb{R}^+\) during limited time

\[
X_{ij}^k = 0 \quad \text{if} \quad W_{ij} > m,
\]

which is equivalent to

\[
X_{ij}^k \geq 0, \quad X_{ij}^k W_{ij} \leq m X_{ij}^k, \quad \forall i, j \in \{1, \ldots, n\}. \tag{3}
\]

For control algorithms with a dynamic region division method, the distance matrix can be generalized to a time dependent matrix \(W^k\) as well.

**Figure 2.** A network flow model of the robust taxi dispatch problem. A circle represents a region with region ID 1, 2, 3, 4. We omit the superscript of time \(k\) since for one time slot only. Uncertain demand is denoted by \(r_i, L_i\) is the original number of vacant taxis before dispatch at region \(i\), and \(X_{ij}\) is a dispatch solution that sending the number of vacant taxis from region \(i\) to region \(j\) with the distance \(W_{ij}\).

One service metric is fairness over all regions, or that the demand-supply ratio of each region equals to that of the whole city. A balanced distribution of vacant taxis is an indication of good system performance from the perspective that a customer’s expected waiting time is short as shown by a queuing theoretic model in [32]. Meanwhile, a balanced demand-supply ratio means that regions with less demand will also get less resources, and idle driving distance will also be reduced in regions with more supply than demand if we pre-allocate possible redundant supply to those regions in need. We define the objective of minimizing demand-supply ratio mismatch between each region and the whole city as minimizing the following function

\[
J_E(X^k, r^k) = \sum_{i} \left( \frac{1}{n} X_{i}^{k} - X_{i}^{k} 1_{n} + L_{i}^{k} \right) \alpha \to 0. \tag{6}
\]

This is because by minimizing (6) under the constraints (4) and (5), we get the same optimal solution of minimizing the following demand-supply ratio mismatch function under constraints (4) and (5).

\[
\sum_{k=1}^{n} \frac{1}{n} X_{i}^{k} - X_{i}^{k} 1_{n} + L_{i}^{k} \to \frac{1}{N}. \tag{7}
\]

It is worth noting that the function \(J_E(X^k, r^k)\) defined as (6) is affine in \(r^k\) for any \(X^k\), and convex in \(X^k\) for any \(r^k\), while the mismatch function (7) is not concave in \(r^k\) for any \(X^k\).

To explain how (6) approximates (7) under constraints (4) and (5), consider the following problem

\[
\text{minimize } b > 0, \sum b_i = c, \quad \sum \frac{a_i}{b_i^{\alpha+1}} = c \quad \text{is a constant.} \tag{8}
\]

Substitute \(b_n = c - b_1 \cdots - b_{n-1}\) into (8), and take partial derivatives of \(\sum a_i b_i^{\alpha+1}\) over \(b_i, i = 1, \ldots, n - 1\). When the minimum of (6) is achieved, each partial derivative should be 0, namely

\[
-\alpha \frac{a_i}{b_i^{\alpha+1}} = -\alpha (-1)^{\alpha} \frac{a_n}{(c - b_1 \cdots - b_{n-1})^{\alpha+1}} = 0,
\]

which is equivalent to

\[
\frac{a_1}{b_1^{\alpha+1}} = \cdots = \frac{a_{n-1}}{b_{n-1}^{\alpha+1}} = \frac{a_n}{b_n^{\alpha+1}}.
\]

Hence, when \(\alpha \to 0, \alpha + 1 \to 1\), the optimal solution of minimizing \(J_E\) over \(X^k\) satisfies

\[
\frac{1}{n} X_{i}^{k} - X_{i}^{k} 1_{n} + L_{i}^{k} \to \frac{1}{N}.
\]
Therefore, with function (3), we map the objective of balancing supply according to demand across every region in the city to a computationally tractable function that concave in the uncertain parameters and convex in the decision variables for a robust optimization problem.

The number of initial vacant taxis $L^k_{j+1}$ depends on the number of vacant taxis at each region after dispatch during time $k$ and the mobility patterns of passengers during time $k$, while we do not directly control the latter. We define $P^k_{ij}$ as the probability that a taxi traverses from region $i$ to region $j$ and turns vacant again (after one or several drop off events) around the beginning of time $k+1$, provided it is vacant at the beginning of time $k$. Examples of getting $P^k_{ij}$ based on data include but not limited to methods of describing trip patterns of taxis [23] and autonomous mobility on demand systems [32].

Then the number of vacant taxis within region $j$ by the end of time $k$ is $(1^T_nX^k - (X^k\mathbf{1}_n)^T + (L^k)^TP^k)$, where $P^k_j$ is the $j$-th column of $P^k$, and

$$(L^k+1)^T = (1^T_nX^k - (X^k\mathbf{1}_n)^T + (L^k)^TP^k).$$

**Weighted-sum objective function:** Since there exists a trade-off between two objectives, we define a weight parameter $\beta$ of two objectives $J_D(X^k)$ in (2) and $J_E(X^k, r^k)$ in (5). Without considering model uncertainties corresponding to $r^k$, a convex optimization form of taxi dispatch problem is

$$\min_{X^k, L^k} \quad J = \sum_{k=1}^{\tau} (J_D(X^k) + \beta J_E(X^k, r^k))$$

s.t (3), (4), (9).

**Robust taxi dispatch problem formulation:** We aim to find out a dispatch solution robust to an uncertain demand model in this work. For time $k = 1, \ldots, \tau$, uncertain demand $r^k$ only affects the dispatch solutions of $k + 1, \ldots, \tau$, similar to the multi-stage robust optimization problem in [9]. Hence, with a list of parameters and variables shown in Table 1, considering effects of current decisions to estimated future costs, a multi-stage robust taxi dispatch problem is defined as following

$$\min \max \min \max \ldots \min \max \quad J = \sum_{k=1}^{\tau} \left( \sum_{i=1}^{N} X_{ij}^k W_{ij} + \beta \frac{r_i^k}{\alpha} \right)$$

s.t $(L^k+1)^T = (1^T_nX^k - (X^k\mathbf{1}_n)^T + (L^k)^TP^k)$,

$$X_{ij}^k W_{ij} \leq m X_{ij}^k,$$

$$X_{ij}^k \geq 0, \quad i, j \in \{1, 2, \ldots, n\}.$$  \quad (11)

After getting an optimal solution $X_{1^{\text{st}}}$ of (11), we adjust the solution by rounding methods to get an integer number of taxis to be dispatched towards corresponding regions. It does not affect the optimality of the result much in practice, since the objective function is related to the demand-supply ratio of each region. A feasible integer solution of (11) always exists, since $X_{ij}^k = 0, \forall i, j, k$ is feasible.

**III. Constructing Uncertainty Sets**

With many factors affecting taxi demand during different time within different areas of a city, explicitly describing the model is a strict requirement and errors of the model will affect the performance of dispatch frameworks. Considering future demand uncertainties benefits for minimizing worst-case demand-supply ratio mismatch error and idle distance described as shown in [22], [23]. However, the uncertainty set constructed by only using a standard deviation range [22], [23] cannot tell how possible the true real-world cost is smaller than the optimal cost. Hence, with a large amount of taxi operational records data, it is essential to construct a model that captures the spatial-temporal demand uncertainties and provides a probabilistic guarantee about the true possible values of costs by solving robust dispatch problem (11).

**A. Samples of concatenated demand vector**

Informally, we consider the concatenated demand vector $r_c$ as a random variable. It is worth noting that we do not have additional assumptions about either the form of $\Delta$ besides closeness and convexity, or the form of marginal distribution of each element of vector $r_c$, or the true distribution of $P^*(r_c)$.

Methods of constructing uncertainty sets in robust optimization literature is typically designed for i.i.d. sampled random vectors that utilize information from a dataset of samples to provide theoretical guarantee for the performance of robust optimization problems [8], [7], [10]. We transform the knowledge of previous work to construct an uncertainty set $\Delta$ for the random vector $r_c$ that contains spatial-temporal relations of the demand model. We assume that one day is discretized as $K$ time slots in total, and the demand of each region during one time slot is described as $r^k, k = 1, \ldots, K$. Then every $\tau$ discretized time slots of $r^k, k = t, t+1, \ldots, t+\tau$ are concatenated to a vector $r_c(t)$ to represent the possible temporal correlations among consecutive time slots. We define one sample of vector $r_c(t)$ of date $d_1$ as $\tilde{r}_c(d_1, t)$, a vector calculated via aggregating total number of pick up events of all taxis at each region for time slots $t, t+1, \ldots, t+\tau$. For instance, for consecutive time slots $(1, \ldots, \tau), (2, \ldots, \tau+1), \ldots$, the sampled vectors on date $d$ for time index $t = 1, 2, \ldots$ are denoted as

$$\tilde{r}_c(d, 1) = \begin{bmatrix} \tilde{r}^1(d) \\ \tilde{r}^2(d) \\ \vdots \\ \tilde{r}^\tau(d) \end{bmatrix}, \quad \tilde{r}_c(d, 2) = \begin{bmatrix} \tilde{r}^{\tau+1}(d) \\ \vdots \\ \tilde{r}^{2\tau}(d) \end{bmatrix}.$$  

We consider demand vectors of different dates for the same time slot as independent samples, i.e., demand $\tilde{r}_c(d_1, t), \tilde{r}_c(d_2, t), \ldots, \tilde{r}_c(d_N, t)$ sampled from $N$ days for time index $t$ are independent with each other for every time index $t$. For convenience, we omit the time index $t$ of $r_c(t)$ in later discussions when there is no confusion.

There are two advantages to building uncertain sets for the concatenated demand model $r_c$. The first one is that theories and results proposed for i.i.d. sampled dataset is applicable to design uncertainty sets based on a spatial-temporal dataset.
The second one is computational efficiency, that we are able to construct an uncertain set with spatial-temporal properties for all regions during several consecutive predicting time slots by calculating a hypothesis testing one time. It is worth noting that the objective function of problem (11), a function concave of $r_k, k = 1, \ldots, \tau$ is still concave of the uncertain parameter $r_c$ with the uncertainty sets constructed in this section. This property guarantees that uncertainty sets constructed in this work can be directly applied for the robust optimization problem (11) with $r^k, k = 1, \ldots, \tau$ as parameters.

B. An uncertainty set with probabilistic guarantee

For convenience, we concisely denote all the variables of the taxi dispatch problem as $x$. Assume that we do not have knowledge about the true distribution $P^*(r_c)$ of the random demand vector $r_c$. When the uncertainty parameter is included in the objective function $J(r_c, x)$ of problem (11), the probabilistic guarantee for the event that the true dispatch cost being smaller than the optimal dispatch cost is described by the following chance constrained problem

$$\min_x \quad M$$
$$\text{s.t} \quad P_{r_c \sim P^*(r_c)}(f(r_c, x) = J(r_c, x) - M \leq 0) \geq 1 - \epsilon. \tag{12}$$

Here $x \in \mathbb{R}^n$ is the optimization variable, and $r_c \in \mathbb{R}^n$ is an uncertain parameter. The constraint $f$ and objective function $J$ are concave in $r_c$ for any $x$, and convex in $x$ for any $r_c$. Without loss of generality about the objective and constraint functions, equivalently we aim to find solutions of the following form of chance constrained problem

$$\min_x \quad J(x)$$
$$\text{subject to} \quad P_{r_c \sim P^*(r_c)}(f(r_c, x) \leq 0) \geq 1 - \epsilon. \tag{13}$$

When it is difficult to explicitly estimate $P^*(r_c)$, given constraints $f(r_c, x)$ that concave in $r_c$ for any $x$, we solve the following robust optimization problem such that optimal solutions of (14) satisfy the probabilistic guarantee of constraints for problem (13)

$$\min_x \max_{r_c \sim \Delta} \quad J(x), \quad \text{subject to} \quad f(r_c, x) \leq 0, \tag{14}$$

Then $r_c$ of problem (14) can be any vector in the uncertainty set $\Delta$ instead of a random vector in problem (13), and we require that by solving an optimization problem with this constrained uncertain set performance of optimal solutions is guaranteed for $r_c \sim P^*$. Another requirement is that the robust optimization problem is computationally tractable problem with this uncertainty set. To emphasize the probability of holding the constraint of (13) with the uncertainty set $\Delta$ of the robust dispatch problem, we denote the uncertainty set as $U_c$ for the the process of constructing a computationally tractable uncertainty set. Hence, for a general form of constraint function $f(r_c, x)$ appeared in robust taxi dispatch problem, the uncertainty set construction problem is defined as the following:

**Problem 1:** Construct an uncertainty set $U_c$, given $\epsilon$ and a data set of random vectors $r_c$, such that (P1). The robust constraint (14) is computationally tractable. (P2). The set $U_c$ implies a probabilistic guarantee for the true distribution $P^*(r_c)$ of a random vector $r_c$ at level $\epsilon$, that is, for any optimal solution $x^* \in \mathbb{R}^n$ and for any function $f(r_c, x)$ concave in $r_c$, we have the implication:

$$\text{If } f(r_c, x^*) \leq 0, \quad \text{for } \forall r_c \in U_c,$$

$$\text{then } P_{r_c \sim P^*(r_c)}(f(r_c, x^*) \leq 0) \geq 1 - \epsilon. \tag{15}$$

The given probabilistic guarantee level $\epsilon$ is related to the degree of conservativeness of the robust optimization problem. The trade-off between the average cost of robust optimal solutions and the probabilistic level is shown by evaluations in Section VI. It is worth noting that a confidence region $U_{c,\epsilon}$ of the random vector that satisfies $P(r_c \in U_{c,\epsilon}) \geq 1 - \epsilon$ does not need to be the same with the uncertainty set $U_c$ satisfies (15) in general [4]. Instead of purely building a confidence region $U_{c,\epsilon}$, we focus on the performance of the robust solutions based on the data-driven uncertainty sets.

The probabilistic guarantee considered in robust optimization literature is stronger than what we require in this work, that the above implication (15) should be satisfied for any feasible solution $x$ of the robust optimization problem [8], [7]. In practice, we will apply the optimal solution of the robust dispatch problems as suggestions for taxi drivers, hence only the optimal solution will affect the performance of the dispatch framework, and we require implication and empirical test of (15) for optimal solutions only in this work.

C. Uncertainty Modeling

In this section, we briefly review the theories related to constructing uncertainty demand models based on a spatial-temporal dataset considered in this work. Since we do not assume that the marginal distribution for every element of vector $r_c$ is independent with each other, we select two approaches without any assumptions about the true distribution $P^*(r_c)$ in the literature [8], [14], [27]. The basic idea is to find a threshold for a hypothesis testing that is acceptable with respect to the given dataset and a required probabilistic guarantee level, and then construct an uncertainty set based on the hypothesis testing. We briefly review the process of a hypothesis testing and the formulations of the uncertainty sets.

Hypothesis testing is a widely applied technique to examine the property of a data set [21]. A hypothesis testing starts from a given null-hypothesis $H_0$ that makes a claim about an unknown distribution $P^*$, and we need to decide whether to accept $H_0$ or reject it, based on a data set $S$ drawn from $P^*$. The fact that a null-hypothesis is false means there is no sufficient evidence to determine its validity. A typical test designs a statistic $T \equiv T(S, H_0)$, and a threshold $\Gamma \equiv \Gamma(\alpha_h, S, H_0)$, where $\alpha_h$ is a given significance level for data set $S$ on hypothesis $H_0$. If $T > \Gamma$, we reject $H_0$. $T$ is also random since it depends on the randomly sampled data $S$. The threshold $\Gamma$ is the value that with a probability at most $\alpha_h$, $H_0$ will be incorrectly rejected with respect to samples $S$.

1) Uncertainty demand sets built from marginal samples:

One intuitive description about a random vector is to define a range for each element.
For instance, David and Nagaraja\cite{14} considered the following multivariate hypothesis with given thresholds $\tilde{q}_{i,0}, \tilde{q}_{i,0} \in \mathbb{R}$, $i = 1, 2, \ldots, \tau_n$

$$
H_{0,i} : \inf \{ t : \mathbb{P}(r_{ci} \leq t) \geq 1 - \frac{\epsilon}{\tau_n} \} \geq \tilde{q}_{i,0} \\
\inf \{ t : \mathbb{P}(\tilde{r}_{ci} \leq t) \geq 1 - \frac{\epsilon}{\tau_n} \} \geq -\tilde{q}_{i,0}.
$$
(16)

This hypothesis is related to the bound of the $\frac{\epsilon}{\tau_n}$ probability value on the random vector, and we divide $\epsilon$ by $\tau_n$ because $r_{ci}$ is a multivariate random vector that we need the hypothesis testing for each component $r_{ci}$ holds simultaneously to provides the probabilistic guarantee described as (15).

Assume that we have $N$ random samples for each component $r_{ci}$ of $r_c$, ordered in increasing value as $r_{c,1}^{(1)}, r_{c,1}^{(2)}, \ldots, r_{c,1}^{(N)}$ no matter the original sample order. Then this order is also the order of the estimated value $\hat{r}_{ci}$, i.e., $\hat{r}_{c,1}^{(1)}, \hat{r}_{c,1}^{(2)}, \ldots, \hat{r}_{c,1}^{(N)}$. We define the index set by

$$
s = \min \left\{ k \in \mathbb{N} : \sum_{j=0}^{N} \left( \frac{1}{\tau_n} \right)^{N-j} (1 - \frac{\epsilon}{\tau_n})^j \leq \frac{\alpha_k}{2\tau_n} \right\},
$$
(17)

and let $s = N + 1$ if the corresponding set is empty. The test $H_{0,i}$ is rejected if

$$
\hat{r}_{c,1}^{(s)} \geq \tilde{q}_{i,0} \text{ or } -\hat{r}_{c,1}^{(N-s+1)} \geq -\tilde{q}_{i,0}.
$$

To construct an uncertainty set, we need an accepted hypothesis test. Hence, we set $\tilde{q}_{i,0} = \hat{r}_{ci}^{(s)}$ and $\tilde{q}_{i,0} = \hat{r}_{ci}^{(N-s+1)}$ with $\hat{r}_{ci}^{(s)}$ and $\hat{r}_{ci}^{(N-s+1)}$ from the sampled dataset, then $H_{0,i}$ is always accepted. The following uncertainty set is then applied in this work based on the hypothesis testing (16).

**Proposition 1 ([8], [27]):** If $s$ defined by equation (17) satisfies that $N - s + 1 < s$, then, with probability at least $1 - \alpha_s$, over the sample, the set

$$
\mathcal{U}_{s}^{c} (r_{ci}) = \left\{ r_{ci} \in \mathbb{R}^{\tau_n} : \hat{r}_{ci}^{(N-s+1)} \leq r_{ci} \leq \hat{r}_{ci}^{(s)} \right\}
$$
(18)

implies a probabilistic guarantee for $\mathbb{P}^* (r_{ci})$ at level $\epsilon$.

The hypothesis (16) is tested for each component $r_{ci}$ separately, and the uncertainty demand model also describes the range of $r_{ci}$, $i = 1, 2, \ldots, \tau_n$ separately provided by Proposition 1. We do not assume that the marginal distributions of $\mathbb{P}^*$ are independent, their correlations are reflected in the box uncertainty set in the sense that changing the value of $n$ and $\tau$ result in a different index value $s$ (17), and the order statistics $\hat{r}_{ci}^{(N-s+1)}$ and $\hat{r}_{ci}^{(s)}$ will be different. However, the model of the box type of uncertainty set formula does not directly describe the spatial-temporal correlations among components of $r_{ci}$.

2) Uncertainty set motivated by moment hypothesis testing: Though the box type of uncertainty set reflects the spatial-temporal correlations by varying range values with different dimensions of $r_{ci}$, it is not easy to tell directly from the uncertainty set (15) when the range of one component changes how will others be affected. To construct an uncertainty set that directly shows the spatial-temporal correlations of the demand model, we consider to apply hypothesis testing related to the first and second moments of the random vector. The following null assumptions are about the mean and covariance of the true distribution $\mathbb{P}^*(r_{ci})$ of random vector $r_{ci}$ (21)

$$
H_0 : \mathbb{E}^\mathbb{P}[r_{ci}] = \mu_0 \text{ and } \mathbb{E}^\mathbb{P}[r_{ci}^2] = \Sigma_0, \text{ with test statistics } T \text{ defined as } ||\hat{r}_{ci} - \mu_0|| \text{ and } \|\hat{\Sigma} - \Sigma_0||. \text{ Given thresholds } \Gamma_1^B \text{ and } \Gamma_2^B, H_0 \text{ is rejected when the difference among the estimate of mean or covariance according to multiple times of samples is greater than the threshold, i.e.,}
$$
$$
||\mathbb{E}^\mathbb{P}[\hat{r}_{ci}] - \hat{\mu}_i|| \geq \Gamma_1^B \text{ or } ||\mathbb{E}^\mathbb{P}[\hat{r}_{ci}^2] - \hat{\Sigma}|| \geq \Gamma_2^B,
$$

where $\mathbb{E}^\mathbb{P}[\hat{r}_{ci}]$ is the estimated mean value of one experiment, $\hat{r}_{ci}$ are the estimated mean and covariance of multiple times of experiments. The remaining problem is then to select the thresholds such that the above hypothesis testing holds given the dataset. In the following Section 4.3 the detailed steps of calculating the thresholds $\Gamma_1^B$ and $\Gamma_2^B$ at a desired significance value $\alpha_h$ and probabilistic guarantee level $\epsilon$ based on the given dataset is described.

The uncertainty set derived based on the moment hypothesis testing is defined in the following proposition.

**Proposition 2 ([8], [27]):** With probability at least $1 - \alpha_h$ with respect to the sampling, the following uncertainty set $\mathcal{U}_{CS}^c (r_{ci})$ provides a probabilistic guarantee level of $\epsilon$ for $\mathbb{P}^* (r_{ci})$

$$
\mathcal{U}_{CS}^c (r_{ci}) = \{ r_{ci} \geq 0, \hat{r}_{ci} + y + CTw : \exists y, w \in \mathbb{R}^{\tau n} \text{ s.t. } ||y||_2 \leq \frac{1}{\sqrt{\epsilon}}, ||w||_2 \leq \sqrt{\frac{1}{\epsilon}} \}
$$
(19)

where $CTC = \hat{\Sigma} + \hat{\Gamma}^B$ is a Cholesky decomposition.

By testing the properties of both first and second moments of the dataset, the uncertainty set (19) reflects the spatial-temporal correlations of the demand model directly compared with the box type (16). When one component of $r_{ci}$ increases or decreases, we have an intuition how it affects the value of other components of $r_{ci}$ by the expression (19). More properties of each type of uncertainty set and application level problems, such as how to choose the number of samples $N$ for the hypothesis testing with high dimensional $r_{ci}$ will be discussed in evaluations of Section 6.

IV. ALGORITHM FOR CONSTRUCTING UNCERTAIN DEMAND SETS

Given a dataset, the algorithm for constructing uncertainty sets includes three main steps—getting a sample set of $r_{ci}$ from the original dataset and partition the sample set, bootstrapping a threshold for the test statistics according to the requirement of the probability guarantee, and calculating the model of uncertainty sets based on the thresholds. In this section, we explain each step, summarize the process in Algorithm 1 and discuss factors to consider for choosing parameters of the algorithm. Numerical examples are shown in Section VI.

A. Aggregating demand and partition the sample set

The first step is to transform the original dataset of taxi operational records to a dataset of sampled vector $\hat{r}_{ci}(d,t)$

2Bootstrapped thresholds and theoretic bounds proposed by work [21] are compared in [8]. The bootstrapped thresholds result in a smaller uncertainty set in general, hence reduces the ambiguity in $\mathbb{P}^*$. In this work, we apply the bootstrapped thresholds $\Gamma_1^B$ and $\Gamma_2^B$ based on the dataset.
of different dates for each index \( t \). For instance, assume we choose the length of each time slot as one hour, and the dataset records all trip information of taxis during each day. According to the start time and GPS coordinates of the pick-up position of each trip, we aggregate the total number of pick-up events during one hour at each region to get samples of \( r^k, k \in \{1, 2, \ldots, \tau\} \) and the concatenated demand vector \( r_c \). It is computationally efficient to process the original data for obtaining a sample set of \( r_c \) in general, though the amount of available taxi trips or trajectory information is large – the time complexity is \( O(N_{\text{record}}) \) of the number of total records \( N_{\text{record}} \). By only passing through the raw data once, we are able to group each pick up and drop off events to a specific discretized time slot and region.

We assume that the dataset contains independent samples of the random vector \( r_c \), and we do not impose any prior knowledge of the true distribution \( P^*(r_c) \). It is always possible to describe the support of the distribution of the entire dataset, even when all samples contained in the dataset do not follow the same distribution, as explained in Figure 3 When there is prior knowledge or categorical information such that the dataset can be partitioned into several subsets according to some feature space, we get a more accurate uncertainty set according to each sub-dataset to provide the same probabilistic guarantee level compared with the uncertainty set from the entire dataset.

Clustering algorithms with categorical information [19] is applicable for dataset partition when information besides pick up events is available in the dataset, such as weather or traffic conditions. It is worth noting that if the uncertainty sets are built for a categorical information set \( I = \{I_1, I_2, \ldots\} \), then for the robust dispatch problems, we require the same set of categories is available in real-time, hence we apply the uncertainty set built for \( I_1 \) to find solutions when the current situation is considered as \( I_1 \). For instance, when there is additional information like weather or traffic condition for each trip provided by the taxi operational records, these types of information can be used as categorical information for clustering. The dataset applied in the evaluations of Section VII does not have additional categorical information of trips that available for a clustering algorithm such as [19], hence, we partition the dataset as demand during weekdays and demand during weekends. Even with this simple and intuitive partition process, we shrink the area of an uncertainty for the same probabilistic guarantee level. Then during weekdays (weekends) we use uncertainty sets built from weekdays (weekends) data to calculate robust dispatch solutions.

B. Algorithm

The uncertainty sets designed in this work require an accepted null hypothesis testing. Given original operational records data, the null hypothesis \( H_0, \alpha_h \), and the test statistics \( T \), we need to find a threshold that accepts \( H_0 \) at significance value \( \alpha \) for each subset of sampled demand vectors. With a threshold of the test statistics calculated via the given dataset, we then apply the formula (18) for constructing a box type of uncertainty set, and the formula (19) for an SOC type of uncertainty set, respectively. The following Algorithm 1 describes the complete process for constructing uncertain demand sets based on the original dataset.

Algorithm 1 Algorithm for constructing uncertain demand sets

**Input:** A dataset of taxi operational records

1. Demand aggregating and sample set partition

   Aggregate demand to get a sample set \( S \) of the random demand vector \( r_c \) from the original dataset. Partition the sample set \( S \) and denote a subset \( S(I_p) \subset S \), \( p = 1, \ldots, P \) as the subset partitioned according to either prior knowledge or categorical information \( I_p \). Denote the partitioned sample subset for each time index \( t \) as \( S(t, I_p) \).

2. Bootstrapping thresholds for test statistics

   for each subset \( S(t, I_p) \) do

   **Initialization:** Testing statistics \( T \), a null-hypothesis \( H_0 \), the probabilistic guarantee level \( \epsilon \), a significance level \( 0 < \alpha_h < 1 \), the number of bootstrap time \( N_B \in \mathbb{Z}_+ \).

   **Estimate the mean \( \hat{r}_c(t, I_p) \) and covariance \( \hat{\Sigma}(t, I_p) \) for vector \( r_c \) based on subset \( S(t, I_p) \).**

   for \( j = 1, \ldots, N_B \) do

   (1) Re-sample \( S^j(t, I_p) = \{ \hat{r}_c(d_1, t, I_p), \ldots, \hat{r}_c(d_N, t, I_p) \} \) data points from \( S(t, I_p) \) with replacement for each \( t \).

   (2) Get the value of the test statistics based on \( S^j(t, I_p) \).

   end for

   (3) Get the thresholds of the \( \alpha \) significance level for \( H_0 \).

   end for

3. Calculate the model of uncertainty sets

   Get the box type and the SOC type of uncertainty sets according to (18) and (19), respectively, for each \( t \) and \( I_p \).

**Output:** Uncertainty sets for problem (11)

We do not restrict the method of estimating mean \( \hat{r}_c(t, I_p) \) and covariance \( \hat{\Sigma}(t, I_p) \) matrices of a subset \( S(t, I_p) \) in step 2, and bootstrap is one method for this step. The estimations of this step are considered as the true mean and covariance for calculating \( \Gamma_1^B \) and \( \Gamma_2^B \) in the following repeated sampling process. For step 2.(2), the process for the box type of uncertainty sets is: calculate index \( s \) that satisfies (17) with the given \( \epsilon \), sort each component of sampled vectors \( r_c(d_t, t, I_p) \), and get the order statistics \( r_{c,i}(N^{-s+1}) \) \( (j_t, I_p) \), \( r_{c,i}(s) \) \( (j_t, I_p) \) of the \( j \)-th sample set \( S^j(t, I_p) \). For the SOC type, we calculate the mean and covariance of the samples of the vector according to the subset \( S^j(t, I_p) \) as \( \hat{r}_c(j_t, I_p) \) and \( \hat{\Sigma}(j_t, I_p) \), respectively.

In step 2.(3), the \( \alpha_h \) level thresholds for the box type of uncertainty sets are the \( \lfloor N_B(1 - \alpha_h) \rfloor \)-th largest value of the
upper bound \(r_c(j, t, I_p)\) and the \([N_B \alpha_h]\)-th largest value of the lower bound \(r_c(N_B+1)(j, t, I_p)\) for the \(i\)-th component of each \(t\) and \(I_p\). For the SOC type of uncertainty sets, we calculate the mean and covariance of \(r_c(j, t, I_p)\) for the \(N_B\) times bootstrap as \(\hat{r}_c(t, I_p)\) and \(\hat{\Sigma}(t, I_p)\), and get
\[
\begin{align*}
\Gamma_1(j, t, I_p) &= ||\hat{r}_c(j, t, I_p) - \hat{r}_c(t, I_p)||_2, \\
\Gamma_2(j, t, I_p) &= ||\hat{\Sigma}(j, t, I_p) - \hat{\Sigma}(t, I_p)||_2.
\end{align*}
\]
Denote the \([N_B(1 - \alpha_h)]\)-th largest value of \(\Gamma_1(j, t, I_p)\) and \(\Gamma_2(j, t, I_p)\) as \(\Gamma_1^B(t, I_p)\) and \(\Gamma_2^B(t, I_p)\), respectively.

Remark 1: The process of constructing uncertainty sets only requires that the hypothesis test is accepted for i.i.d. samples of the random vector. We accept the hypothesis test when there is not enough evidence to reject it, which does not mean the claim of \(H_0\) is true. This property is very important for constructing the uncertainty demand set of the robust dispatch problem, since the true distribution function of a demand model can be complex and we only have datasets of taxi operational records instead of ground truth knowledge of the distribution function. Hence, even without enough knowledge of the true, high-dimensional demand model, based on the dataset and an accepted hypothesis test, we are able to construct an uncertainty set with probabilistic guarantee for the robust taxi dispatch problem.

It is worth noting that the above Algorithm 1 provides a valid estimation of uncertain sets based on hypothesis testing and bootstrapped thresholds for the robust resource allocation problem when the sampled data set is consistent with the real world scenario. For demand missed in the dataset, for instance, some customers might leave the request queue after waiting for a long time and the operational records did not show the event of picking up the customer, we are not able to get the exact rate of missed customers. However, missed requests are only part of the historical requests, and this type of events is also random – for instance, even for the same time length of waiting, some customers were more patient and finally got a taxi. By constructing an uncertainty set to describe the demand model based on occurred records of the original dataset, we involve the effect of random missing events better than only applying a deterministic model from this perspective.

In summary, to construct a spatial-temporal uncertain demand model for the robust taxi dispatch problem, in this section, we consider the taxi operational record of each day as one independent and identically distributed (i.i.d.) sample for the concatenated demand vector \(r_c\). By partitioning the entire dataset to several subsets according to categorical information such as weekdays and weekends, we are able to build uncertainty sets for each subset of data without additional assumptions about the true distribution of the spatial-temporal demand profile. Then we apply theories proved for i.i.d. samples of random vectors in the literature [8] [14] [27] to construct a box type and an SOC type of uncertainty sets. The key advantage of the data-driven approach we propose is that we do not rely on prior knowledge of the true distribution of the random demand vector to provide a desired probabilistic guarantee of robust solutions. Furthermore, theories proved for i.i.d. datasets are applicable to construct uncertainty sets that reflect the spatial-temporal correlations of the demand model.

V. COMPUTATIONALLY TRACTABLE FORMULATIONS

We build equivalent computationally tractable formulations of problem (11) with different definitions of uncertain sets built in Section III in this section, and show that the robust taxi dispatch problem in this work can be solved efficiently. Computational tractability of a robust linear programming problem for ellipsoid uncertainties are discussed in [5]. The process is to reformulate constraints of the original problem to equivalent convex constraints that must hold given the uncertainty set. The objective function of problem (11) is concave of the uncertain parameters \(r^k\), convex of the decision variables \(X^k, L^k\) with the decision variables on the denominators, hence, not standard forms of linear programming (LP) or semi-definite programming (SDP) problems that already covered by previous work [5]. Hence, we prove one equivalent computationally tractable form of problem (11) for each uncertainty set constructed in Section III.

Only the \(J_E\) components of objective functions in (11) include uncertain parameters, and the decision variables of the function are in the denominator of the function \(J_E\). The box type uncertainty set defined as (18) is a special form of polytope, hence, we first prove an equivalent standard form of convex optimization problem for (11) for a polytope uncertainty set as the following.

Theorem 1: (Next step dispatch) If the uncertainty set of problem (11) when \(\tau = 1\) is defined as the following polytope
\[
\Delta := \{r \geq 0, Ar \leq b\}
\]
and we omit the superscripts \(k\) for variables and parameters without confusion. Then problem (11) with \(\tau = 1\) is equivalent to the following convex optimization problem

\[
\begin{align*}
\text{minimize} & \quad \sum_i \sum_j X_{ij} W_{ij} + b^T \lambda \\
\text{subject to} & \quad A^T \lambda - \beta \begin{bmatrix} \frac{1}{(\Gamma_1^B)^2} \left( \frac{1}{\Gamma_1^B J^2} \cdot X_{i1} + L_{i1} \right)^2 & \cdots & \frac{1}{(\Gamma_1^B)^2} \left( \frac{1}{\Gamma_1^B J^2} \cdot X_{in} + L_{in} \right)^2 \end{bmatrix} \geq 0, \\
& \quad 1^T \lambda X - XL_{in} + L^T \geq 1, \\
& \quad X_{ij} W_{ij} \leq m_{ij}, \\
& \quad X_{ij} \geq 0, \quad \forall i, j \in \{1, \ldots, n\}.
\end{align*}
\]

Proof: See Appendix VIII-A.

For the multi-stage robust optimization problem (11), we prove that the order of minimize and maximum is exchangeable in the following theorem, and equivalent computationally tractable forms are proved based on this theorem.

Lemma 1: (Exchange the order of minimize and maximum) Assume that the definition of the uncertainty set \(\Delta\) satisfies that the domain of each \(r^k\) is a compact set, then the multi-stage robust dispatch problem defined as (11) is equivalent to the following robust dispatch problem

\[
\begin{align*}
\min_{X^k, L^k} & \quad \max_{r^k \in \Delta_k} J = \sum_{k=1}^\tau (J_D(X^k) + \beta J_E(X^k, r^k)) \\
\text{s.t.} & \quad \text{constraints of (11), } k = 1, \ldots, \tau.
\end{align*}
\]
Here $L^1$ is the initial number of vacant taxis within each region before dispatch provided by sensor information, not a decision variable, and we omit the time index of $L^k, k = 2, \ldots, \tau$ in minimization for notation convenience.

Proof: See Appendix VIII-B

For the multi-stage robust optimization problem (11), the computationally tractable convex form depends on the definition of uncertainty sets. For a multi-stage robust optimization problem that minimax theorem does not hold, an approximated semidefinite programming form for calculating time dependent control input of linear dynamical systems affected by uncertainty is proposed in [9]. When conditions of Lemma 1 hold, equivalent convex optimization forms of problem (11) are derived based on problem (21).

The box type uncertainty set (18) is a special form of polytope, that the uncertain demand model during different time of a day is described separately. The process of converting the one-stage robust optimization problem (11) to an equivalent computationally tractable convex form is similar to that of the one-stage robust optimization problem. The result is described as the following lemma.

**Lemma 2:** If the uncertain set for $r^k, k = 1, \ldots, \tau$ describes each demand vector $r^k$ separately as a polytope with the form

$$
\Delta_k := \{r^k \geq 0, A_k r^k \leq b_k\}, \quad k = 1, \ldots, \tau,
$$

problem (11) is equivalent to the following convex optimization problem

$$
\begin{align*}
\min_{\tau, (X^k, L^k, z) \geq 0} & \quad \sum_{k=1}^{\tau} \left( \sum_{i,j} X^k_{ij} W_{ij} + b_k^T \lambda^k \right) \\
\text{subject to} & \quad A_k^T \lambda^k - \beta \begin{bmatrix}
\frac{1}{\tau} (X^k_{n1} - \bar{X}_{n1} L_n^k)^T \\
\vdots \\
\frac{1}{\tau} (X^k_{n\tau} - \bar{X}_{n\tau} L_n^k)^T
\end{bmatrix} \geq 0, \\
\text{other constraints of (11), } & \quad k = 1, \ldots, \tau.
\end{align*}
$$

Proof: See Appendix VIII-C1

For a more general case that the uncertainty sets for $r^1, \ldots, r^\tau$ are temporally correlated, the following theorem and proof describe the equivalent computationally tractable convex form of (11).

**Theorem 2:** When $\Delta$ is defined as the following polytope

$$\Delta := \{(\Delta_1, \ldots, \Delta_\tau) | A_1 r^1 + \cdots + A_\tau r^\tau \leq b, r^k \geq 0\},$$

problem (11) is equivalent to the following convex optimization problem

$$
\begin{align*}
\min_{\tau, (X^k, L^k, \lambda) \geq 0} & \quad \sum_{k=1}^{\tau} \left( \sum_{i,j} X^k_{ij} W_{ij} + b_k^T \lambda \right) \\
\text{subject to} & \quad A_k^T \lambda - \beta \begin{bmatrix}
\frac{1}{\tau} (X^k_{n1} - \bar{X}_{n1} L_n^k)^T \\
\vdots \\
\frac{1}{\tau} (X^k_{n\tau} - \bar{X}_{n\tau} L_n^k)^T
\end{bmatrix} \geq 0, \\
\text{other constraints of (11), } & \quad k = 1, \ldots, \tau.
\end{align*}
$$

Proof: See Appendix VIII-C2

With an uncertain demand model defined as [19] for concatenated $r^1, \ldots, r^\tau$, the following theorem derive the equivalent computationally tractable form of problem (11).

**Theorem 3:** When the uncertainty set for $r^1, \ldots, r^\tau$ is defined as the SOC form of [19], problem (11) is equivalent to the following convex optimization problem (26).

$$
\begin{align*}
\min_{\tau, (X^k, L^k, z)} & \quad \sum_{k=1}^{\tau} \sum_{i,j} X^k_{ij} W_{ij} \\
& \quad + \beta \left( \hat{r}_c^T z + \Gamma_1^B \|z\|_2 + \sqrt{\frac{1}{\epsilon} - 1} \|Cz\|_2 \right) \\
\text{subject to} & \quad c_l(X) \leq z, \\
& \quad \text{constraints of (11), } \quad k = 1, \ldots, \tau,
\end{align*}
$$

where $c_l(X) \in \mathbb{R}^{\tau n}$ is the concatenation of $c(X^1), \ldots, c(X^\tau)$.

Proof: See Appendix VIII-D

It is worth noting that any optimal solution for problem (10) has a special form between any pair of regions $(i, q)$.

**Proposition 3:** Assume $X_1^1, \ldots, X_\tau^\tau$ is an optimal solution of (10), then any $X^k_{\ast}$ satisfies that for any pair of $(p, q)$, at least one value of the two elements $X^k_{pq}$ and $X^k_{qp}$ is 0.

Proof: We prove by contradiction. Assume that one optimal solution has the form $X^k_{pq} > 0$ such that $X^k_{pi} > 0$ and $X^k_{qi} > 0$. Without loss of generality, we assume that $X^k_{qi} \geq X^k_{pq}$, and let

$$X^k_{pq} = X^k_{pq} + X^k_{pq} = 0,$$

other elements of $X^k_{\ast}$ equal to $X^k$. Then

$$1_n^T X^k_i = 1_n^T X^k_i + 1_n^T = 1_n^T X^k_i - X^k_{pi} + X^k_{pq} = 0,$$

because

$$\sum_{i\neq j} X^k_{ij} - \sum_{l\neq q} X^k_{il} = X^k_{ij} + \sum_{l\neq q} X^k_{il} \neq X^k_{ij},$$

Hence, we have $J_E(X^k_{\ast}, r^k) = J_E(X^k_{\ast}, r^k)$. All constraints are satisfied and $X^k_{\ast}$ is also a feasible solution for (11).

Next, we compare $J_D(X^k_{\ast})$ and $J_D(X^k_{\ast})$. With $X^k_{pq} > X^k_{pi} > 0$ and $X^k_{pq} = X^k_{pi} - X^k_{pi} \geq 0$, we have

$$X^k_{pi} > X^k_{pi} = X^k_{pi} - X^k_{pi} \geq 0,$$

Thus the partial cost $J_D(X^k_{\ast}) > J_D(X^k_{\ast})$, which contradicts with the assumption that $X^k$ is an optimal solution. To summarize, we show that an optimal solution cannot have $X^k_{pi} > 0, X^k_{pi} > 0$ at the same time, and at least one of $X^k_{pi}$ and $X^k_{pi}$ should be 0.

With equivalent convex optimization forms under different uncertainty sets, robust taxi dispatch problem (11) is computationally tractable and solved efficiently.

**VI. DATA-DRIVEN EVALUATIONS**

We conduct data-driven evaluations based on four years of taxi trip data of New York City [15]. A summary of this data set is shown in Table IV. In this data set, every record represents an individual taxi trip, which includes the GPS coordinators
of pick up and drop off locations, and the date and time (with precision of seconds) of pick-up and drop-off locations. One region partition example according to the map of Manhattan of New York City is shown in Figure 4, where we visualize the density of taxi passenger demand with data we used for our large-scale data-driven evaluation. The lighter the region, the higher the daily demand density. As we can see in the figure, the middle regions typically have higher density than the uptown and downtown regions in Manhattan. We construct uncertainty sets according to Algorithm 1, discuss factors that affect modeling of the uncertainty set, and compare optimal costs of the robust dispatch formulation (11) and the non-robust optimization form (10) in this section.

How vacant taxis are balanced across regions with different $\alpha$ values: Figure 5 shows mismatch between supply and demand defined as (7) for different optimal solutions of minimizing $J_E$ defined in (6) for $\alpha \in (0, 1]$. With $\alpha$ closer to 0, the optimal value of (7) is smaller. We choose $\alpha = 0.1$ for calculating optimal solutions of (11) and (10) in this section.

### A. Box type of uncertainty set

For all box type of uncertainty sets shown in this subsection with the model described in Subsection III-C1, we set the confidence level of hypothesis testing as $\alpha_h = 10\%$, bootstrap time as $N_b = 1000$, number of randomly sampled data (with replacement) for each time of bootstrap as $N = 10000$.

**Partitioned dataset compared with non-partitioned dataset:** We show the effects of partitioning the trip record dataset by weekdays and weekends in Figure 6 and 7. The whole city is partitioned into 50 regions and the prediction time horizon $\tau = 4, \epsilon = 0.3$, and every $r_c \in \mathbb{R}^{200 \times 1}$. Figures 6 and 7 show the lower and upper bounds of each region during one time slot of 18. By applying data of weekdays and weekends separately, the range $[\hat{r}_{c,i}^{(s)}, \hat{r}_{c,i}^{(N-s+1)}]$ of each component is reduced. To get a measurement of the uncertainty level, we defined the sum of range of every component for $\hat{r}_c$ as the following

$$U(\hat{r}_c) = \sum_{s=1}^{N} (\hat{r}_{c,i}^{(s)} - \hat{r}_{c,i}^{(N-s+1)})$$

For the box type of uncertainty sets, when values of the dimension of $r_c$, i.e., $\tau n$, $\alpha_h$ and $\epsilon$ are fixed, a smaller $U(\hat{r}_c)$ means a smaller area of the uncertainty set, or a more accurate model. We denote $U(\hat{r}_c)$ calculated via records of weekdays and weekends as $U_{wd}(\hat{r}_c)$ and $U_{wn}(\hat{r}_c)$ respectively, compared with $U(\hat{r}_c)$ constructed from the complete dataset, we have

$$\frac{U(\hat{r}_c) - U_{wd}(\hat{r}_c)}{U(\hat{r}_c)} = 52\%,$$  
$$\frac{U(\hat{r}_c) - U_{wn}(\hat{r}_c)}{U(\hat{r}_c)} = 28\%.$$  

This result shows that when by constructing an uncertainty set for each subset of partitioned data, we reduce the range of uncertainty sets to provide the same level of probabilistic guarantee for the robust dispatch problem. This is because samples contained in each subset of data do not follow the same distribution and can be categorized as two clusters.

**Choose an appropriate $N$ for high-dimensional $r_c$:** It is worth noting that the index $s$ affects the range selection for every component $r_{c,i}$, hence, for different values of $\alpha_h, \epsilon, \tau, n$, we should adjust the number of samples $N$ to get an accurate estimation of the marginal range. As shown in Table V, $N$ need to be large enough for a large $\tau n$ value, or $s$ is too close to $N$ and the upper and lower bounds $\hat{r}_{c,i}^{(N-s+1)}, \hat{r}_{c,i}^{(s)}$ cover almost the whole range of samples. Hence, the box type uncertainty set is not a good choice for large $\tau n$ value, though the computational cost of solving problem (25) is smaller than that of (26) with the same size of $\tau n$.  

![Map of Manhattan area in New York City.](image1)  

![Demand and supply mismatch value change with $\alpha$.](image2)  

![Comparison of demand and supply mismatch values defined as (7) with different solutions for minimizing $J_E$ defined in (6) with $\alpha$ in range (0, 1]. The value of function (7) under an optimal solution of $J_E$ is smaller with an $\alpha$ closer to 0, which means the dispatch solution tends to be more balanced throughout the entire city.](image3)  

![Comparison of box type of uncertainty sets constructed from all data and those constructed only based on trip records of weekdays. When keeping all parameters the same, by applying data of weekdays, the range of uncertainty set for each $r_{c,i}$ is smaller than that based on the whole dataset.](image4)  

![Comparison of box type of uncertainty sets constructed from all data and uncertainty sets constructed only based on trip records of weekends.](image5)
### Table II

**NEW YORK CITY DATA IN THE EVALUATION SECTION.**

<table>
<thead>
<tr>
<th>Taxi Trip Data set</th>
<th>Format</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collection Period</td>
<td>Data Size</td>
</tr>
<tr>
<td>01/01/2010-12/31/2013</td>
<td>about 7 million</td>
</tr>
</tbody>
</table>

### Table III

**VALUE OF INDEX s FOR THE BOX TYPE UNCERTAINTY SET (1). FOR LARGE τn, N need to be large, or s is too close to N that the range covers values of almost all samples.**

<table>
<thead>
<tr>
<th>N</th>
<th>αk</th>
<th>ε</th>
<th>n</th>
<th>τ</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>10000</td>
<td>0.1</td>
<td>0.2</td>
<td>50</td>
<td>2</td>
<td>9992</td>
</tr>
<tr>
<td>10000</td>
<td>0.1</td>
<td>0.5</td>
<td>50</td>
<td>2</td>
<td>9970</td>
</tr>
<tr>
<td>10000</td>
<td>0.3</td>
<td>0.2</td>
<td>50</td>
<td>2</td>
<td>9991</td>
</tr>
<tr>
<td>10000</td>
<td>0.1</td>
<td>0.2</td>
<td>1000</td>
<td>2</td>
<td>9999</td>
</tr>
<tr>
<td>10000</td>
<td>0.1</td>
<td>0.5</td>
<td>1000</td>
<td>2</td>
<td>9999</td>
</tr>
</tbody>
</table>

### Table IV

**COMPARING THRESHOLDS WITH AND WITHOUT DISCRIMINATING WEEKDAYS AND WEEKENDS DATA. WHEN Γ^B_1 OR Γ^B_2 IS SMALLER, THE VOLUME OF THE UNCERTAINTY SET IS SMALLER. HERE n = 1000, τ = 3, N = 10000, ε = 0.3, αk = 0.2.**

<table>
<thead>
<tr>
<th>Data type</th>
<th>Weekdays</th>
<th>Weekends</th>
<th>Non partitioned</th>
</tr>
</thead>
<tbody>
<tr>
<td>Γ^B_1</td>
<td>10.53</td>
<td>13.84</td>
<td>17.96</td>
</tr>
<tr>
<td>Γ^B_2</td>
<td>2576.94</td>
<td>2923.35</td>
<td>3864.47</td>
</tr>
</tbody>
</table>

### Table V

**COMPARING THRESHOLDS OF SOC UNCERTAINTY SETS FOR DIFFERENT DIMENSIONS r_c, BY CHANGING EITHER THE REGION PARTITION NUMBER n OR THE PREDICTION TIME HORIZON τ.**

<table>
<thead>
<tr>
<th>Γ^D</th>
<th>Γ^D_1</th>
<th>Γ^D_2</th>
<th>Γ^D_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 50, τ = 1</td>
<td>42.37</td>
<td>1.52 \times 10^6</td>
<td></td>
</tr>
<tr>
<td>n = 50, τ = 3</td>
<td>52.68</td>
<td>4.29 \times 10^6</td>
<td></td>
</tr>
<tr>
<td>n = 50, τ = 6</td>
<td>107.35</td>
<td>8.25 \times 10^6</td>
<td></td>
</tr>
<tr>
<td>n = 10, τ = 3</td>
<td>71.35</td>
<td>3.36 \times 10^6</td>
<td></td>
</tr>
<tr>
<td>n = 1000, τ = 3</td>
<td>10.53</td>
<td>2576.94</td>
<td></td>
</tr>
</tbody>
</table>

### B. SOC type of uncertainty set

The SOC type of uncertainty set is a high-dimensional convex set that is not able to be plotted. The bootstrapped thresholds for the hypothesis testing to construct the SOC uncertainty sets based on partitioned and non-partitioned data are summarized in Table IV. **Similarly as the box type of uncertainty sets, when we separate the dataset and construct an uncertainty demand model for weekdays and weekends respectively, the sets are smaller compared to the uncertain demand model for all dates.** When α and ε values are fixed, with smaller Γ^B_1 and Γ^B_2, the demand model UC^CS is more accurate to guarantee that with at least probability 1 – ε, the constraints of the robust dispatch problems are satisfied. Numerical results of this conclusion are shown in Table IV.

**How n and τ affect the accuracy of uncertainty sets:**

For a box type of uncertainty set, when τn is a large value, the bootstrap sample number N should be large enough such that index s is not too close to N. Without a large enough sample set, we choose to construct an SOC type of uncertainty set (such as τn = 1000, N = 10000 in Table V). Since SOC captures more information about the second moment properties of the random vector compared with the box type uncertainty set, some uncorrelated components of r_c will be reflected by the estimated covariance matrix, and the volume of the uncertainty set will be reduced. We show the value of Γ^B_1 and Γ^B_2 with different dimensions of r_c or τn values in Table IV. When increasing the value of τn, values of Γ^B_1 and Γ^B_2 are reduced, which means the uncertainty set is smaller. However, it is not helpful to reduce the granularity of region partition to a smaller than street level, since we construct the model for a robust dispatch framework and a too large n is not computationally efficient for the dispatch algorithm.

### C. Compare robust solutions with non-robust solutions

For testing the quality of the uncertainty sets applied in the robust dispatch problems, we use the idea of cross-validation from machine learning. The dataset is separated as a training set for building the uncertain demand model, and a testing set for comparing the results of the dispatch solutions. The customer demand models applied in the robust and non-robust optimization problems are different. For the non-robust dispatch problem, the demand prediction r^k is a deterministic value. For instance, in this work we use the average or mean of the bootstrapped value of the training dataset. In the experiments, the idle geographical distance of one taxi between a drop-off event of one passenger and the following pick-up event is approximately as the one norm distance between the 2D geographical coordinates (provided as longitude and latitude values of GPS data in the trip dataset) of the two points. Then the corresponding idle miles on ground is converted from the geographical distance according to the geographical coordinates of New York City.

In the robust dispatch problem, the part that directly includes the uncertain demand r^k is the penalty function for violating a balanced demand-supply ratio requirement. For each testing data r^k, we denote the demand-supply ratio mismatch error of a dispatch solution as C. We then compare the value of C of robust dispatch solutions with the SOC type of uncertainty set constructed in this work with the value of C of non-robust solutions of testing samples. The distribution of values are shown in Figure. The average demand-supply ratio error is reduced by 31.7% with robust solutions. We compare the cost distribution of total idle distance in Figure. It shows the average total idle distance is reduced by 10.13%. For all testing, the robust dispatch solutions result in no idle distance greater than 0.8 \times 10^5, and non-robust solutions has 48% of samples with idle distance greater than 0.8 \times 10^5. The cost of robust dispatch is a weighted sum of both the demand-supply ratio error and estimated total idle driving distance, and the average cost is reduced by 11.8% with robust solutions. It means that the performance of the system is improved when the true demand deviates from the average historical value considering model uncertainty information in the robust dispatch process. It is worth noting that the number of total idle distance shown in this figure is the direct calculation result of the robust dispatch problem. When we convert the number to an estimated value of corresponding miles in one year, the result is a total reduction of 20 million miles in NYC.

**Check whether the probabilistic level ε is guaranteed:**
Theoretically, the optimal solution of the robust dispatch problems with the uncertainty set should guarantee that with at least the probability \((1 - \epsilon)\), when the system applies the robust dispatch solutions, the actual dispatch cost under a true demand is smaller than the optimal cost of the robust dispatch problem. Figures 10 and 11 show the cross-validation testing result that the probabilistic guarantee level is reached for both box type and SOC type of uncertainty sets via solving (25) and (26), respectively. Comparing these two figures, one key insight is that the robust dispatch solution with an SOC type uncertainty set provides a tighter bound on the probabilistic guarantee level that can be reached under the true random demand compared with solutions of the box type uncertainty set. It shows the advantage of considering second order moment information of the random vector, though the computational cost is higher to solve problem (26) than to solve problem (25).

**How probabilistic guarantee level affects the average cost:** There exists a trade-off between the probabilistic guarantee level and the average cost with respect to a random vector \(r_c\). Selecting a value for \(\epsilon\) is case by case, depending on whether a performance guarantee for the worst case scenario is more important or whether the average cost performance is more important. For a high probabilistic guarantee level or a large \(1 - \epsilon\) value, the average cost may not be good enough since we minimize a worst case that rarely happens in the real world. When the \(1 - \epsilon\) value is relatively small, the average cost can also be large since many possible values of the random vector are not considered.

We compare the optimal cost of robust solutions and average cost of empirical tests for two types of uncertainty sets via solving (25) and (26) in Figure 12 and 13 respectively. The
optimal cost of the robust dispatch framework shows that the result of minimized worst case scenario for all possible $r_c$ included in the uncertainty set, and the average cost of empirical tests show the real world scenario when we applying the optimal solution to dispatch taxis under random demand $r_c$. The horizontal line shows the average cost of non-robust solutions since this cost is not related to $\epsilon$. The $\epsilon$ values that provide the best average costs are not exactly the same for different types of uncertainty sets according to the experiments. For the box type of uncertainty set shown in Figure $\epsilon = 0.3$ provides the smallest average experimental cost, and for SOC type of uncertainty set shown in Figure $\epsilon = 0.25$ provides the smallest average experimental cost. The minimum average cost of an SOC robust dispatch solution is smaller than that of a box type. It indicates that the second order moment information of the random variable should be included for modeling the uncertainty set and calculating robust dispatch solutions for the dataset we use in this section, though its computational cost is higher.

VII. Conclusion

In this paper, we develop a multi-stage robust optimization model considering demand model uncertainties in taxi dispatch problems. We model spatial-temporal correlations of the uncertainty demand by partitioning the entire data set according to categorical information, and applying theories without assumptions on the true distribution of the random demand vector. We prove that an equivalent computationally tractable form exist with the constructed polytope and SOC types of uncertainty sets, and the robust taxi dispatch solutions are applicable for a large-scale transportation system. A robust dispatch formulation that purely minimizes the worst-case cost under all possible demand usually sacrifices the average system performance. The robust dispatch method we design allows any probabilistic guarantee level for a minimum cost solution, considering the trade-off between the worst-case cost and the average performance. Evaluations show that under the robust dispatch framework we design, the average demand-supply ratio mismatch error is reduced by 31.7%, and the average total idle driving distance is reduced by 10.13% or about 20 million miles in total in one year. In the future, we will enhance problem formulation considering more uncertain characteristics of taxi network model, like traffic conditions.

REFERENCES

The domain of problem (29) satisfies that optimal solution, i.e., exchange the order of max and min without changing such an that is also optimal for the maximin problem, and we can nonempty. It means there exists an optimal minimax solution parameters are compact, the set of saddle points of (29) is compact, and the domain of each $r^k$ is closed and convex, i.e., is compact, and Lemma 1 holds. Considering the maximizing part of problem (31)
\[
\max_{r^k \in \Delta_k} J, \quad \text{s.t. constraints of (11)},
\]
the Lagrangian of (30) with multipliers $\lambda^k \geq 0, v^k \geq 0$ is
\[
\mathcal{L}(X^k, r^k, \lambda^k, v^k) = \sum_{k=1}^{\tau} (J_D(X^k) + b_k^T \lambda^k - (A_k^T \lambda^k - c(X^k) - v^k)^T r^k),
\]
Hence, based on the proof of Theorem 1, we take partial derivative of the Lagrangian (31) for every $r^k \in \Delta_k$. Since strong duality holds for problem (30), an equivalent form of (11) under uncertainty set (22) is defined as (23).

2) Proof of Theorem 2: Proof: With uncertain set defined as (24), the domain of each $r^k$ is compact and Lemma 1 holds. We consider the equivalent problem (21) of problem (11), and first derive the Lagrangian of the maximum part of the objective function (21) with constraint $\lambda \geq 0, v_k \geq 0$
\[
\mathcal{L}(X^k, r^k, \lambda, v_k)
\]
Similarly as the proof of Theorem 1 we take the partial derivative of (32) over each $r^k$, the objective function of the dual problem is
\[
g(X^k, L^k, \lambda, r^k) = \sup_{r^k \in \Delta_k} \mathcal{L}(X^k, r^k, \lambda, v_k)
\]
Since strong duality holds, problem (25) is a equivalent to the computationally tractable convex optimization form (11) under uncertain set (24).

D. Proof of Theorem 3

Proof: Under the definition of uncertainty set (19) for concatenated $r^k$, the domain of each $r^k$ is compact, and problem (11) is equivalent to (21). We now consider the dual form for the objective function $\sum_{k=1}^{\tau} J_E(X^k, r^k)$ that relates to $r^k$. By the definition of inner product, we have
\[
\sum_{k=1}^{\tau} c^T(X^k)r^k = c_1^T(X)r_1, c_1(X) = [c^T(X^1) \ldots c^T(X^\tau)]^T.
\]
When the uncertainty set of \( r_c \) is an SOC defined as (19), problem (21) is equivalent to

\[
\begin{align*}
\min_{X^k, L^k} \max_{r_c \geq 0} & \quad \left( c_l^T(X)r_c + \sum_{k=1}^{\tau} \sum_{i} \sum_{j} X_{ij}^k W_{ij} \right) \\
\text{subject to} & \quad r_c = \hat{r}_c + y + C^Tw, \\
& \quad \| y \|_2 \leq \Gamma_B^1, \| w \|_2 \leq \sqrt{\frac{1}{\epsilon} - 1}, \\
& \quad \text{constraints of (11)}. 
\end{align*}
\] (33)

We first consider the following minimax problem related to the uncertainty set

\[
\begin{align*}
\max_{r_c \geq 0} & \quad c_l^T(X)r_c \\
\text{subject to} & \quad r_c = \hat{r}_c + y + C^Tw, \\
& \quad \| y \|_2 \leq \Gamma_B^1, \| w \|_2 \leq \sqrt{\frac{1}{\epsilon} - 1}. 
\end{align*}
\] (34)

To get the dual form of problem (34), we start from the following Lagrangian with \( v \geq 0 \)

\[
L(X, r_c, z, v) = c_l^T(X)r_c + z^T(\hat{r}_c + y + C^Tw - r_c) + v^Tr_c.
\]

By taking the partial derivative of the above Lagrangian over \( r_c \), we get the supreme value of the Lagrangian as

\[
\sup_{r_c} L(X, r_c, z, v) = \begin{cases} 
\hat{r}_c^Tz + \Gamma_B^1 \| z \|_2 + \sqrt{\frac{1}{\epsilon} - 1} \| Cz \|_2, & \text{if } c_l(X) \leq z \\
\infty, & \text{o.w.} 
\end{cases}
\]

Then with the norm bound of \( y \) and \( w \), we have

\[
\sup_{\| y \|_2 \leq \Gamma_B^1, \| w \|_2 \leq \sqrt{\frac{1}{\epsilon} - 1}} (z^T(\hat{r}_c + y + C^Tw)) = \hat{r}_c^Tz + \Gamma_B^1 \| z \|_2 + \sqrt{\frac{1}{\epsilon} - 1} \| Cz \|_2.
\]

Hence, the objective function of the dual problem for (34) is

\[
g(X, r_c, z) = \sup_{r_c \in U^C} L(X, r_c, z) = \begin{cases} 
\hat{r}_c^Tz + \Gamma_B^1 \| z \|_2 + \sqrt{\frac{1}{\epsilon} - 1} \| Cz \|_2, & \text{if } c_l(X) \leq z \\
\infty, & \text{o.w.} 
\end{cases}
\]

Together with the objective function \( J_D(X^k) \) and other constraints that do not directly involve \( r_c \), an equivalent convex form of (11) given the uncertainty set (19) is shown as (26).