Problem 1: Symbolizing

Provide a logic translation for each of the following.

1. Every sorting algorithm that is asymptotically faster than mergesort is limited in what kinds of elements can be in its list.
   - Domain: algorithms
     - \( S(x) \): \( x \) is a sorting algorithm
     - \( M(x) \): \( x \) is faster than mergesort
     - \( U(x) \): works on lists of any kind
     
     \[ \forall x . (S(x) \land M(x)) \rightarrow \neg U(x) \]

2. I love everyone who loves me as long as they also like peanut butter or cheddar cheese on their lemon sorbet.
   - Domain: people
     - \( L(x, y) \): \( x \) loves \( y \)
     - \( P(x) \): \( x \) likes peanut butter on lemon sorbet
     - \( C(x) \): \( x \) likes cheddar cheese on lemon sorbet
     - \( m \): Me

     \[ \forall x . (L(x, m) \land (P(x) \lor C(x))) \rightarrow L(m, x) \]

What more practice? See Practice Quiz 03 for a list from our textbooks

Problem 2: Prosify

Convert the following proof outlines into prose proofs.

3. Theorem: \( (P \land Q) \rightarrow R \equiv P \rightarrow (R \lor \neg Q) \)
   - Proof outline:
     \[ (P \land Q) \rightarrow R \equiv \neg(P \land Q) \lor R \equiv (\neg P \lor \neg Q) \lor R \equiv \neg P \lor (\neg Q \lor R) \equiv \neg P \lor (R \lor \neg Q) \equiv P \rightarrow (R \lor \neg Q) \]
   - Proof:
     \( (P \land Q) \rightarrow R \) is equivalent to \( \neg(P \land Q) \lor R \) by the definition of implication; De Morgan’s law changes that to \( (\neg P \lor \neg Q) \lor R \), which is the same as \( \neg P \lor (R \lor \neg Q) \) by the associative and commutative properties of disjunction. Using the definition of implication again, we arrive at \( P \rightarrow (R \lor \neg Q) \).
4. Theorem: There is an integer that every other integer divides.
Formalism: \( \exists x . \forall y . D(x,y) \) where \( D(x,y) \) means “\( x \) divides \( y \)”.
Proof outline: \( x = 1; y ÷ 1 = y \) remainder 0; \( : \exists x . \forall y . D(x,y) \)

Proof.
Consider \( x = 1 \). 1 divides \( y \) with no remainder regardless of what \( y \) is. Thus, \( \forall y . D(1,y) \). Since this works for \( x = 1 \), we know that \( \exists x . \forall y . D(x,y) \).

\[ \square \]

What more practice? Take any proof from Quiz 02 or our textbook and try converting to prose

Problem 3 Complete

Fill in the blanks to complete the following proofs by cases.

Theorem 1 \((P \land Q) \to M \equiv P \to (Q \to M)\)

Proof. Either \( P \) is true or it is false.

Case 1: \( P \) is true The expression \((P \land Q) \to M\) in this case

6. can be simplified to \( Q \to M \) by the equivalence of \( \top \land Q \) and \( Q \).

The expression \( P \to (Q \to M) \) in this case

7. can be simplified to \( Q \to M \) by the equivalence of \( \top \to Q \) and \( Q \).

Because the two are equivalent to the same thing, they are equivalent to each other.

Case 2: \( P \) is false The expression \((P \land Q) \to M\) in this case

8. can be simplified to \( \bot \to M \) by the equivalence of \( \bot \land Q \) and \( \bot \), which in turn is just \( \top \) regardless of the value of \( M \).

The expression \( P \to (Q \to M) \) in this case

9. is \( \bot \to (Q \to M) \), which is \( \top \) regardless of the values of \( Q \) and \( M \).

Because the two are equivalent to the same thing, they are equivalent to each other.

Since \((P \land Q) \to M \equiv P \to (Q \to M)\) is true in both cases, it is true in general. \( \square \)
Theorem 2 \( P \oplus Q \equiv \overline{P} \oplus \overline{Q} \)

Proof. 10. Either \( P \) is true or it is false.

Case 1: \( P \) is true  The expression \( P \oplus Q \) in this case

11. is \( T \oplus Q \), which is defined to mean \( (T \lor Q) \land (T \land \overline{Q}) \).
Simplifying, that is equivalent to \( T \land \overline{Q} \), or simply \( \overline{Q} \).

The expression \( \overline{P} \oplus \overline{Q} \) in this case

12. is \( \bot \oplus Q \), which is defined to mean \( (\bot \lor Q) \land (\bot \land \overline{Q}) \).
We can simplify that to \( (\overline{Q}) \land (\bot) \),
which is equivalent to \( \overline{Q} \land \top \) or simply \( \overline{Q} \).

Because the two are equivalent to the same thing, they are equivalent to each other.

Case 2: \( P \) is false  The expression \( P \oplus Q \) in this case

13. is \( \bot \oplus Q \), which is defined to mean \( (\bot \lor Q) \land (\bot \land \overline{Q}) \).
Simplifying, that is equivalent to \( Q \land \top \), or simply \( Q \).

The expression \( \overline{P} \oplus \overline{Q} \) in this case

14. is \( T \oplus \overline{Q} \), which is defined to mean \( (T \lor \overline{Q}) \land (T \land Q) \).
Simplifying, that is equivalent to \( T \land \overline{Q} \), which is equivalent to \( \overline{Q} \).

Because the two are equivalent to the same thing, they are equivalent to each other.

Because \( P \oplus Q \equiv \overline{P} \oplus \overline{Q} \) is true in both cases, it is true in general. \( \Box \)

What more practice? Try MCS problem 1.7; writing the example proofs in \( \forall x \) 15.6, 16.3, 19.2, and 19.6 in prose; \( \forall x \) practice 15.A and 15.B