CS 2102 - DMT1 - Spring 2020 — Luther Tychonievich
Practice exercise in class friday february 28, 2020

Practice 06

problem 1 *Convert to prose*

Convert the following symbolic proof that `f(x)` = x^2 to prose.

```
let f(x) be computed as
    if x <= 0 then return 0
    else return (2*x-1) + f(x-1)
```

Symbolic Proof.

1		$f(0) = 0 = 0^2$	definition
2	2	$f(x-1) = (x-1)^2$	assumption
	3	$f(x) = 2x - 1 + f(x-1)$	definition
	4	$f(x) = 2x - 1 + (x-1)^2$	combine line 2 and 3
	5	$f(x) = 2x - 1 + (x^2 - 2x + 1)$	algebra on line 4
	6	$f(x) = x^2$	simplify line 5
3		$\forall x \geq 0 \,.\, f(x) = x^2$	principle of induction on lines 1 and 2

Proof.

We proceed by induction on `x`.

Base case: Assume `x` = 0. Then `f(x)` = 0, which is `x*x`.

Inductive step: Assume that `x` > 0 and that `f(x-1) = (x-1)*(x-1)`. Then `f(x) = 2*x-1 + f(x-1)`; replacing `f(x-1)` with `(x-1)*(x-1)` gives us `f(x)` = `2*x - 1 + (x-1)*(x-1)`. Distributing the last term gives us `2*x - 1 + (x*x - 2*x + 1)` which can be simplified to `x*x`.

By the principle of induction, it follows that `f(x)` always returns `x*x`. □

PROBLEM 2 *Code termination*

Prove by induction that each of the following functions terminate given any integer argument.

2.
```
let f(x) be computed as
    if x <= 0 then return x
    otherwise return 1 + f(x-1)
```

Proof.

We proceed by induction on x.

Base case: Assume $x \leq 0$. Then the function terminates immediately by taking the first branch of the `if` statement.

Inductive step: Assume that $x > 0$ and `f(x-1)` terminates. Then the function takes the second branch of the `if` statement and terminates after invoking `f(x-1)` and performing one addition.

By the principle of induction, it follows that `f(x)` terminates for all integer x. □

3.
```
let f(x) be computed as
    if x <= 1 then return x
    otherwise return 1 + f(x-1) + f(x-2)
```

Proof.

We proceed by induction on x.

Base case: Assume $x \leq 1$. Then the function terminates immediately by taking the first branch of the `if` statement.

Inductive step: Assume that $x > 1$ and that `f(y)` terminates for all $y < x$. Then the function takes the second branch of the `if` statement, invoking both `f(x-1)` and `f(x-2)`. Since both `x-1` and `x-2` are $< x$, both invocations terminate, so `f(x)` also terminates.

By the principle of induction, it follows that `f(x)` terminates for all integer x. □

4.
```
let f(x) be computed as
    if x >= -1 then return x
    otherwise return 1 + f(x+1)
```

Proof.

We proceed by induction on x.

Base case: Assume $x \geq -1$. Then the function terminates immediately by taking the first branch of the `if` statement.

Inductive step: Assume that $x < -1$ and that `f(x+1)` terminates. Then the function takes the second branch of the `if` statement and terminates after invoking `f(x+1)` and performing one addition.

By the principle of induction, it follows that `f(x)` terminates for all integer x. □

PROBLEM 3 *Code property*

Prove by induction each of the following functions returns an even number given any non-negative integer argument.

5.
```
let f(x) be computed as
    y = 0
    repeat x times:
        y += 2
    return y
```

Proof.

We proceed by induction on x.

Base case: Assume $x = 0$. Then the loop runs 0 times, meaning $f(x) = 0$, which is even.

Inductive step: Assume that $x > 0$ and that `f(x-1)` is even. Then the loop runs one more time than it did for `f(x-1)`, meaning y is 2 larger than `f(x-1)`'s return value. Since `f(x-1)` is even and 2 is even and the product of two even numbers is even, y must also be even meaning `f(x)` is even.

By the principle of induction, it follows that `f(x)` always returns an even number. □

6.
```
let f(x) be computed as
    if x <= 0 then return 0
    else return 4 * f(x-1)
```
Proof.

We proceed by induction on x.

Base case: Assume $x = 0$. Then $f(x) = 0$, which is even.

Inductive step: Assume that $x > 0$ and that $f(x-1)$ is even. Then $f(x) = 4 + f(x-1)$; since 4 is even and $f(x-1)$ is even and the sum of two even numbers is even, $f(x)$ must also be even.

By the principle of induction, it follows that $f(x)$ always returns an even number. □

7.
```
let f(x) be computed as
    if x <= 0 then return 2
    else return 2 * f(x-2)
```
Proof.

We proceed by induction on x.

Base case: Assume $x = -1$[*]. Then $f(x) = 2$, which is even.

Assume $x = 0$. Then $f(x) = 2$, which is even.

Inductive step: Assume that $x > 0$ and that $f(x-2)$ is even. Then $f(x) = 2 \times f(x-2)$; since 2 is even and $f(x-2)$ is even and the product of two even numbers is even, $f(x)$ must also be even.

By the principle of induction, it follows that $f(x)$ always returns an even number. □

[*] Note that -1 is not in the domain of the proof (which is non-negative integers), but but will be reached by the code when given a non-negative integer (such as 1) so it is a valid and important base case.