**PROBLEM 1 Convert to prose**

Convert the following symbolic proof that \( f(x) = x^2 \) to prose.

1. let \( f(x) \) be computed as
   - if \( x \leq 0 \) then return 0
   - else return \( (2x-1) + f(x-1) \)

**Symbolic Proof.**

<table>
<thead>
<tr>
<th>Line</th>
<th>Expression</th>
<th>Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( f(0) = 0 )</td>
<td>( 0^2 )</td>
</tr>
<tr>
<td>2</td>
<td>( f(x-1) = (x-1)^2 )</td>
<td>( (x-1)^2 )</td>
</tr>
<tr>
<td>3</td>
<td>( f(x) = 2x - 1 + f(x-1) )</td>
<td>( 2x - 1 + (x-1)^2 )</td>
</tr>
<tr>
<td>4</td>
<td>( f(x) = 2x - 1 + (x-1)^2 )</td>
<td>( f(x) = 2x - 1 + (x^2 - 2x + 1) )</td>
</tr>
<tr>
<td>5</td>
<td>( f(x) = 2x - 1 + (x^2 - 2x + 1) )</td>
<td>( f(x) = x^2 )</td>
</tr>
<tr>
<td>6</td>
<td>( f(x) = x^2 )</td>
<td>( f(x) = x^2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \forall x \geq 0 . f(x) = x^2 )</td>
<td>principle of induction on lines 1 and 2</td>
</tr>
</tbody>
</table>

**Proof.**

We proceed by induction on \( x \).

**Base case:** Assume \( x = 0 \). Then \( f(x) = 0 \), which is \( x \times x \).

**Inductive step:** Assume that \( x > 0 \) and that \( f(x-1) = (x-1) \times (x-1) \). Then \( f(x) = 2 \times x - 1 + f(x-1) \); replacing \( f(x-1) \) with \( (x-1) \times (x-1) \) gives us \( f(x) = 2 \times x - 1 + (x-1) \times (x-1) \). Distributing the last term gives us \( 2 \times x - 1 + (x \times x - 2 \times x + 1) \) which can be simplified to \( x \times x \).

By the principle of induction, it follows that \( f(x) \) always returns \( x \times x \). \( \blacksquare \)
**Problem 2 Code termination**

Prove by induction that each of the following functions terminate given any integer argument.

2. let \( f(x) \) be computed as
   
   \[
   \text{if } x \leq 0 \text{ then return } x \\
   \text{otherwise return } 1 + f(x-1)
   \]

   *Proof.*

   We proceed by induction on \( x \).

   **Base case:** Assume \( x \leq 0 \). Then the function terminates immediately by taking the first branch of the if statement.

   **Inductive step:** Assume that \( x > 0 \) and \( f(x-1) \) terminates. Then the function takes the second branch of the if statement and terminates after invoking \( f(x-1) \) and performing one addition.

   By the principle of induction, it follows that \( f(x) \) terminates for all integer \( x \). \( \square \)

3. let \( f(x) \) be computed as
   
   \[
   \text{if } x \leq 1 \text{ then return } x \\
   \text{otherwise return } 1 + f(x-1) + f(x-2)
   \]

   *Proof.*

   We proceed by induction on \( x \).

   **Base case:** Assume \( x \leq 1 \). Then the function terminates immediately by taking the first branch of the if statement.

   **Inductive step:** Assume that \( x > 1 \) and \( f(y) \) terminates for all \( y < x \). Then the function takes the second branch of the if statement, invoking both \( f(x-1) \) and \( f(x-2) \). Since both \( x-1 \) and \( x-2 \) are \( < x \), both invocations terminate, so \( f(x) \) also terminates.

   By the principle of induction, it follows that \( f(x) \) terminates for all integer \( x \). \( \square \)
4. let f(x) be computed as
   if x >= -1 then return x
   otherwise return 1 + f(x+1)

Proof.

We proceed by induction on x.

**Base case:** Assume x ≥ –1. Then the function terminates immediately by taking the first branch of the if statement.

**Inductive step:** Assume that x < –1 and that f(x+1) terminates. Then the function takes the second branch of the if statement and terminates after invoking f(x+1) and performing one addition.

By the principle of induction, it follows that f(x) terminates for all integer x. □

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**Problem 3 Code property**

Prove by induction each of the following functions returns an even number given any non-negative integer argument.

5. let f(x) be computed as
   y = 0
   repeat x times:
      y += 2
   return y

Proof.

We proceed by induction on x.

**Base case:** Assume x = 0. Then the loop runs 0 times, meaning f(x) = 0, which is even.

**Inductive step:** Assume that x > 0 and that f(x-1) is even. Then the loop runs one more time than it did for f(x-1), meaning y is 2 larger than f(x-1)’s return value. Since f(x-1) is even and 2 is even and the product of two even numbers is even, y must also be even meaning f(x) is even.

By the principle of induction, it follows that f(x) always returns an even number. □
6. let \( f(x) \) be computed as
   \[
   \begin{align*}
   &\text{if } x \leq 0 \text{ then return 0} \\
   &\text{else return } 4 \times f(x-1)
   \end{align*}
   \]
   
   Proof.

   We proceed by induction on \( x \).

   Base case: Assume \( x = 0 \). Then \( f(x) = 0 \), which is even.

   Inductive step: Assume that \( x > 0 \) and that \( f(x-1) \) is even. Then \( f(x) = 4 + f(x-1) \); since 4 is even and
   \( f(x-1) \) is even and the sum of two even numbers is even, \( f(x) \) must also be even.

   By the principle of induction, it follows that \( f(x) \) always returns an even number. \( \square \)

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7. let \( f(x) \) be computed as
   \[
   \begin{align*}
   &\text{if } x \leq 0 \text{ then return } 2 \\
   &\text{else return } 2 \times f(x-2)
   \end{align*}
   \]
   
   Proof.

   We proceed by induction on \( x \).

   Base case: Assume \( x = -1 \). Then \( f(x) = 2 \), which is even.

   Assume \( x = 0 \). Then \( f(x) = 2 \), which is even.

   Inductive step: Assume that \( x > 0 \) and that \( f(x-2) \) is even. Then \( f(x) = 2 \times f(x-2) \); since 2 is even and
   \( f(x-2) \) is even and the product of two even numbers is even, \( f(x) \) must also be even.

   By the principle of induction, it follows that \( f(x) \) always returns an even number. \( \square \)

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* Note that \(-1\) is not in the domain of the proof (which is non-negative integers), but will be reached by the code when given a non-negative integer (such as 1) so it is a valid and important base case.