Practice 07

Problem 1  Products and Powers

Write out the following in full.

1. \( \{1, 2\} \times \{3\} \times \{1, 4\} = \{(1, 3, 1), (1, 3, 4), (2, 3, 1), (2, 3, 4)\} \)

2. \( \{56\}^3 = \{(56, 56, 56)\} \)

3. \( \{1, 2\} \times \mathcal{P}(\{1\}) = \{(1, \{}), (1, \{1\}), (2, \{}), (2, \{1\})\} \)

   \( ((1, 2, \{}), (1, 2, \{(1, 2, 3, 4)\})) \); some uses will treat singleton sets as their
   one element to get \( ((1, 2, \{}), (1, 2, 1, 2, 3, 4) \) or chose not to flatten to get
   \( ((1, 2), \{}), ((1, 2), \{(1, 2, 3, 4)\}) \) instead.

4. \( \{1, 2\} \times \mathcal{P}((1, 2, 3, 4)) = \{(1, 2), \{(1, 2, 3, 4)\}, \{(1, 2, 3, 4)\}, \{(1, 2, 3, 4)\}\} \)

   \( \text{or} ((1, 2), \{(1, 2, 3, 4)\}) \text{ instead.} \)

5. \( \{a, b\}^2 = \{“aa”, “ab”, “ba”, “bb”\} \)

6. \( \{4, 1\} \times \{1, 2\} = \{(4, 1), (4, 2), (1, 1), (1, 2)\} \)

7. \( \{4\} \times \{1, 2\} \times \{3\}^3 = \{(4, 1, 3, 3, 3), (4, 2, 3, 3, 3)\} \)

8. \( \mathcal{P}(\{\})^2 = \{\{\}, \{\}\} \)

Problem 2  Members of Products and Powers

Give two different example members of each of the following sets. Make them different from one another:

different lengths, different internal patterns, etc., is the set allows that. If there are not enough elements of
the set to give two different elements, leave some blanks blank.

9. \( \{a, b, c\}^4 \text{ contains } \) _______ “aaaa” _______ and _______ “cbac” _______

10. \( \{a, b, c\}^1 \text{ contains } \) _______ “a” _______ and _______ “c” _______

11. \( \{a, b, c\}^0 \text{ contains } \) _______ “” _______ and _______ “” _______

12. \( \{a, b, c\}^* \text{ contains } \) _______ “ba” _______ and _______ “cababac” _______
13. \( \text{“good”, “fun”) }^2 \) contains \text{“goodgood”} and \text{“fungood”}

Give two strings of length 3 belonging to

14. \( \{\text{“a”, “ok”}\}^* \): \text{“aaa”} and \text{“aok”}

15. \( \{\text{“a”, “bb”, “ccc”}\}^* \): \text{“bba”} and \text{“ccc”}

**Problem 3 Subsequences**

**Definition 1** A subsequence is a sequence that can be derived from another sequence by deleting zero or more elements without changing the order of the remaining elements.

What are the subsequences of the string “OK”? \text{“”, “O”, “K”, “OK”}

What is the longest subsequence shared by “MATHEMATICS” and “COMPUTERS”? \text{“MTES”}

**Problem 4 Summation proofs**

Prove the following theorems by induction.

16. \( \forall n \in \mathbb{N}, \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2} \)

Proof.

We proceed by induction.

**Base Case** When \( n = 0 \) we have \( \sum_{i=0}^{0} i = 0 \) and \( \frac{(0)(1)}{2} = 0 \), so the theorem holds for \( n = 0 \).

**Inductive step** Assume the theorem holds for some \( n \in \mathbb{N} \): that is, \( \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2} \). Adding \( n + 1 \) to both sides, we have

\[
\sum_{i=0}^{n+1} i = n + 1 + \sum_{i=0}^{n} i = n + 1 + \frac{(n)(n+1)}{2} \]

the left-hand side is equivalent to \( \sum_{i=0}^{n+1} i \) by the definition of summation; the right-hand side can be rearranged using algebra to get

\[
\frac{2(n+1) + (n)(n+1)}{2} = \frac{(2+n)(n+1)}{2} = \frac{(n+1)((n+1)+1)}{2} \]

this means that \( \sum_{i=0}^{n+1} i = \frac{(n+1)((n+1)+1)}{2} \), or in other words that the theorem holds for \( n + 1 \).

By the principle of induction, the theorem holds for all \( n \in \mathbb{N} \). \( \square \)
17. \( \forall n \in \mathbb{N}. \sum_{x=0}^{n} \frac{1}{2^x} = \frac{2^{n+1} - 1}{2^n} \)

Proof.

We proceed by induction.

**Base Case**  When \( n = 0 \) we have \( \sum_{x=0}^{0} \frac{1}{2^x} = 1 \) and \( \frac{2^1 - 1}{2^0} = 1 \), so the theorem holds for \( n = 0 \).

**Inductive step**  Assume the theorem holds for some \( n \in \mathbb{N} \): that is, \( \sum_{x=0}^{n} \frac{1}{2^x} = \frac{2^{n+1} - 1}{2^n} \). Adding \( \frac{1}{2^{n+1}} \) to both sides, we have \( \sum_{x=0}^{n} \frac{1}{2^x} + \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}} + \frac{2^{n+1} - 1}{2^n} \); the left-hand side is equivalent to \( \sum_{x=0}^{n+1} \frac{1}{2^x} \) by the definition of summation; the right-hand side can be rearranged to get \( \frac{1 + 2(2^{n+1} - 1)}{2^{n+1}} = \frac{2^{n+2} - 1}{2^{n+1}} \); this means that \( \sum_{x=0}^{n+1} \frac{1}{2^x} = \frac{2^{n+2} - 1}{2^{n+1}} \), or in other words that the theorem holds for \( n + 1 \).

By the principle of induction, the theorem holds for all \( n \in \mathbb{N} \). \( \square \)
18. \( \forall n \in \mathbb{N} . \ \sum_{x=n}^{2n} x = \frac{3(n+1)n}{2} \)

Proof.

We proceed by induction.

**Base Case**  When \( n = 0 \) we have \( \sum_{x=0}^{0} 0 = 0 \) and \( \frac{3(0)9}{2} = 0 \), so the theorem holds for \( n = 0 \).

**Inductive step**  Assume the theorem holds for some \( n \in \mathbb{N} \): that is, \( \sum_{x=n}^{2n} x = \frac{3(n+1)n}{2} \). Consider the sum evaluated at \( n + 1 \):

\[
\sum_{x=n+1}^{2(n+1)} x = -n + 2n + 1 + 2n + 2 + \sum_{x=n}^{2n} x
\]

\[
= 3n + 3 + \sum_{x=n}^{2n} x
\]

\[
= 3n + 3 + \frac{3(n+1)n}{2}
\]

\[
= 3n + 3 + \frac{3n^2 + 3n}{2}
\]

\[
= 6n + 6 + 3n^2 + 3n
\]

\[
= \frac{3(n^2 + 3n + 2)}{2}
\]

\[
= \frac{3(n+2)(n+1)}{2}
\]

\[
= \frac{3((n+1)+1)(n+1)}{2}
\]

which means the theorem holds at \( n + 1 \) as well.

By the principle of induction, the theorem holds for all \( n \in \mathbb{N} \). \( \Box \)
19. \( \forall x \in \{a \mid a \in \mathbb{Z} \land a \geq -1\} . \sum_{k=-1}^{x} 12 - 2k = 26 + 11x - x^2 \)

Proof.

We proceed by induction.

Base Case When \( x = -1 \) we have \( \sum_{k=-1}^{-1} 12 - 2k = 14 = 26 - 11 - 1 \), so the theorem holds for \( x = -1 \).

Inductive step Assume the theorem holds for some \( x \); that is, \( \sum_{k=-1}^{x} 12 - 2k = 26 + 11x - x^2 \). Consider the sum evaluated at \( x + 1 \):

\[
\sum_{k=-1}^{x+1} 12 - 2k = 12 - 2(x + 1) + \sum_{k=-1}^{x} 12 - 2k
= 10 - 2x + 26 + 11x - x^2
= (11 - 1) - 2x + 26 + 11x - x^2
= 26 + (11 + 11x) - (1 + 2x + x^2)
= 26 + 11(x + 1) - (x + 1)^2
\]

which means the theorem holds at \( x + 1 \) as well.

By the principle of induction, the theorem holds for all \( x \in \{a \mid a \in \mathbb{Z} \land a \geq -1\} \). \( \square \)
You might also try doing inductive proofs with other summation formulae, such as

\[
\sum_{i=0}^{n} i^2 = \frac{(n+1)(2n+1)(n)}{6}
\]

\[
\sum_{i=1}^{n+1} i^2 = \frac{(n+2)(2n+3)(n+1)}{6}
\]

\[
\sum_{i=2}^{n+2} i^2 = \frac{(n+3)(2n+5)(n+2)}{6}
\]

\[
6 \sum_{i=0}^{n} i^3 - i = \left( \frac{n+2}{4} \right)
\]

\[
\sum_{x=0}^{n} \frac{x^2 - 1}{x+1} = \frac{(n+1)(n-1)}{2}
\]

\[
\sum_{x=0}^{n} x^3 - x^2 = \frac{(n+1)(3n+2)(n)(n-1)}{12}
\]

\[
\sum_{i=0}^{n} 3i^2 + 2i = \frac{(2n+3)(n+1)(n)}{2}
\]

\[
\sum_{x=n}^{n} x = \frac{n+n^4}{2}
\]

\[
\sum_{x=0}^{2n} (-1)^x x = n
\]

\[
\sum_{i=1}^{n} \frac{1}{2^i} = \frac{2^n - 1}{2^n}
\]

\[
\sum_{k=-n}^{0} k = \frac{(n+1)n}{-2}
\]

\[
\sum_{i=1}^{n} \frac{1}{3i} = \frac{3^n - 1}{3^n2}
\]

\[
\forall k \neq 1 . \left( \sum_{i=1}^{n} \frac{1}{k^i} = \frac{k^n - 1}{k^n(k-1)} \right)
\]

Note: at least one of the above formulae is false. In the process of proving it you should find the normal methods not working, revealing the non-truth.