Practice 09

Your may answer any question with factorial, choose, and unresolved arithmetic notation, but may not use ellipses. For example, the following are all OK: $\boxed{120}$, $\boxed{5!}$, $\boxed{\frac{5\cdot4\cdot3\cdot2\cdot1}{(2\cdot1)(3\cdot2\cdot1)}}$, $\boxed{\binom{5}{3}}$; however, the following is *not* OK: $\boxed{10\cdot9\cdot8\cdots2\cdot1}$.

PROBLEM 1 *Stand-alone problems*

1. $\binom{21}{8}$ How many 8-element subsets of a 21-element set are there?

2. 2^5 How many strictly-increasing sequences of the numbers $\{1,2,3,4,5\}$ are there?

3. $5\times10^9-5\approx5\times10^9$ My passphrase is a six-word extract taken randomly from the 5-billion-word string created by concatenating all Wikipedia articles. If no six-word string is repeated twice in that corpus, how many passwords can be created in this method?

4. $\frac{26!}{18!}\approx6.3\times10^{10}$ My passphrase is an eight-character string made up of a random collection of lower-case letters (from the 26 letters `a` through `z`), without repeating any letter. How many passwords can be created in this method?

5. $26^8\approx2\times10^{11}$ My passphrase is an eight-character string made up of a random collection of lower-case letters (from the 26 letters `a` through `z`), allowing letter repetitions. How many passwords can be created in this method?

6. $4\cdot6-3=21$ I roll four fair six-sided dice and total the result. How many possible numbers could I roll?

7. $\frac{1}{6^4}=\frac{1}{1296}$ I roll four fair six-sided dice and total the result. What is the chance the total will be 4?

8. * $\frac{6^2+2(5^2+4^2+3^2+2^2+1^2)}{6^4}$ I roll four fair six-sided dice and total the result. What is the chance the total will be 14?

9. $\binom{52}{5}$ I draw five cards from a deck of 52 distinct cards. How many distinct hands of cards could I get?

*Note: will not appear in a quiz as we did not cover this material in spring 2020

10. $8!$ ______ How many ways of shuffling a list of 8 distinct numbers are there?

11. $\dfrac{8!}{1! \cdot 2! \cdot 3! \cdot 2!}$ ______ How many permutations of the sequence $(\perp, \top, \top, \emptyset, \emptyset, \emptyset, \lambda, \lambda)$ are there?

12. 7776^6 ______ My passphrase is six random words taken from a list of 7776 unique words. If I allow words to be repeated, how many passphrases can be created in this method?

13. $\dfrac{7776!}{7770!}$ ______ My passphrase is six random words taken from a list of 7776 unique words. If I do not allow words to be repeated, how many passphrases can be created in this method?

14. $8 \cdot 2 - 1 = 15$ ______ I roll two fair eight-sided dice and total the result. How many possible totals could I roll?

15. $\dfrac{1+1+1}{8^2}$ ______ I roll two fair eight-sided dice and total the result. What is the chance the total will be 4?

16. $\dfrac{20+20}{20+20+17}$ ______ I have a bag of 20 cyan balls, 20 yellow balls, and 20 magenta balls. I took three out, all the same color, and gave them away. If I reach in randomly and draw another ball, what is the chance it will be a different color than the first three?

17. $\dfrac{20+20}{20+20+20}$ ______ I have a bag of 20 cyan balls, 20 yellow balls, and 20 magenta balls. I took three out, all the same color, then put them back in. If I reach in randomly and draw another ball, what is the chance it will be a different color than the first three?

18. $\dfrac{1}{1000}$ ______ A special lottery lets you pick a 3-digit number (leading 0s, like 000 and 023, are allowed); one number, determined but not revealed when the lottery was created, causes you to win \$100 if you are the first person to pick it. 500 people have picked numbers so far (you don't know what they picked) and none have won. If you pick the next number, what is the chance you'll win the \$100?

You might also try the following from past semesters' finals, which do not have a released key.

19. ______________________________ A seven-character computing ID is 3 letters, 1 digit, and 3 more letters. All 26 letters are used, but digits are limited to 2 through 9 (no 0 or 1). How many seven-character computing ID can this scheme create?

20. ______________________________ How many 6-element subsets of a 10-element set are there?

21. ______________________________ Which is larger: $\binom{45}{10}$ or $\binom{45}{40}$?

22. ______________________________ How many 6-element sequences can be made from elements of a 50-element set without repeating elements?

23. ______________________________ How many 6-element sequences can be made from elements of a 50-element set where no element can appear twice in a row? For example, $(1, 2, 1, 2, 1, 2)$ is OK, but $(1, 2, 2, 1, 2, 1)$ is not OK.

24. ______________________________ If I randomly shuffle a list containing 10 `d`s and 16 `x`s, what is the probability the shuffle will result in the exact sequence `"ddddddddddxxxxxxxxxxxxxxxx"`?

25. ______________________________ In a fair raffle, every participant has an equal chance of winning. I participate in two fair raffles: one with 10 people (myself included), one with 100 (myself included). What is my chance of winning at least one raffle?

26. ______________________________ Which adds more options when constructing sequences: doubling the number of options for each spot in the sequence or doubling the length of the sequence? Answer with one of **options**, **length**, or **same**. You may assume both the options and length are initially at least 2.

27. ______________________________ An economy license plate number starts with X or W, then two more letters (out of 24 options, not 26, because O and I are not used), then four digits (all ten used). Repetition is allowed (e.g., "WWX 0000" is OK). How many license plate numbers can this scheme create?

28. ______________________________ How many 9-element subsets of a 100-element set are there?

29. ______________________________ I draw 3 cards from a deck of 50 distinct cards and line them up in a row. How many distinct rows of cards could I get?

30. ____________________________ Which is larger: $\binom{45}{40}$ or $\binom{45}{42}$?

31. ____________________________ A palindrome is a string is the same if you reverse it, like "rrynyrr". How many 7-letter palindromes can be made from the set of 26 letters?

32. ____________________________ I randomly shuffle a list containing five `"d"`s, five `"q"`s, and five `"w"`s,. What is the probability the shuffle will result in the exact sequence `"dqwdqwdqwdqwdqw"`?

33. ____________________________ I randomly shuffle a 15-item list containing the integers 1 through 15. What is the probability the shuffle will result in the exact sequence $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15)$?

34. ____________________________ I roll a pair of twenty-sided dice, with sides numbered 1 through 20. What is the probability at least one die will roll a 6?

35. ____________________________ You can either choose a 5-letter string (out of 26 letters, which can repeat; e.g `"xyxxy"` or `"dfghj"` are both options) or a 5-digit number (out of ten digits, which can repeat, but the first digit must not be 0; e.g. `21020` is an option but `02102` is not). How many options do you have?

PROBLEM 2 *Problems about Bogosort*

Bogosort sorts a list be shuffling it, checking to see if it is in order, and then shuffling again if not. We have two versions: version **R** shuffles randomly each time; version **U** other shuffles in a way that guarantees each shuffling will be unique (i.e., it never checks the same permutation twice).

36. $\frac{1}{20!}$ If given a list of 20 distinct numbers, what is the chance **R** will get the sorted list after just one shuffle?

37. $1 \div \binom{20}{10}$ If given a list of 20 numbers consisting of ten 1s and ten 2s, what is the chance **R** will get the sorted list after just one shuffle?

38. $\frac{2}{20!}$ If given a list of 20 distinct numbers, 0 through 18 with 0 repeated in the list twice; what is the chance **U** will get the sorted list after just one shuffle?

39. $\frac{1}{20!} + \frac{20! - 1}{20!^2} + \frac{(20! - 1)^2}{20!^3}$ How likely **R** to get the right answer after no more than three tries given a list of 20 distinct numbers?

40. $\frac{3}{20!}$ How likely is **U** to get the right answer after no more than three tries given a list of 20 distinct numbers?

41. $n!$ If I know nothing about the contents of the list, but know it contains n values, how many times could **U** shuffle the list in the worst case before it gets the list sorted?

42. all values distinct

(continuing from the previous problem) Describe that worst-case list.

PROBLEM 3 *Summation proofs*

Prove the following theorems by induction.

43. $\forall n \in \mathbb{N} \,.\, \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$

Proof.

We proceed by induction.

Base Case When $n = 0$ we have $\sum_{i=0}^{0} i = 0$ and $\frac{(0)(1)}{2} = 0$, so the theorem holds for $n = 0$.

Inductive step Assume the theorem holds for some $n \in \mathbb{N}$: that is, $\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$. Adding $n+1$ to both sides, we have $n + 1 + \sum_{i=0}^{n} i = n + 1 + \frac{(n)(n+1)}{2}$; the left-had side is equivalent to $\sum_{i=0}^{n+1} i$ by the definition of summation; the right-hand side can be rearranged using algebra to get $\frac{2(n+1)+(n)(n+1)}{2} = \frac{(2+n)(n+1)}{2} = \frac{(n+1)((n+1)+1)}{2}$; this means that $\sum_{i=0}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$, or in other words that the theorem holds for $n+1$.

By the principle of induction, the theorem holds for all $n \in \mathbb{N}$. $\square$

44. $\forall n \in \mathbb{N} \,.\, \sum_{x=0}^{n} \frac{1}{2^x} = \frac{2^{n+1}-1}{2^n}$

Proof.

We proceed by induction.

Base Case When $n = 0$ we have $\sum_{x=0}^{0} \frac{1}{2^x} = 1$ and $\frac{2^1 - 1 = 1}{2^0 = 1} = 1$, so the theorem holds for $n = 0$.

Inductive step Assume the theorem holds for some $n \in \mathbb{N}$: that is, $\sum_{x=0}^{n} \frac{1}{2^x} = \frac{2^{n+1} - 1}{2^n}$. Adding $\frac{1}{2^{n+1}}$ to both sides, we have $\frac{1}{2^{n+1}} + \sum_{x=0}^{n} \frac{1}{2^x} = \frac{1}{2^{n+1}} + \frac{2^{n+1} - 1}{2^n}$; the left-had side is equivalent to $\sum_{x=0}^{n+1} \frac{1}{2^x}$ by the definition of summation; the right-hand side can be rearranged to get $\frac{1 + 2(2^{n+1} - 1)}{2^{n+1}} = \frac{2^{n+2} - 1}{2^{n+1}}$; this means that $\sum_{x=0}^{n+1} \frac{1}{2^x} = \frac{2^{n+2} - 1}{2^{n+1}}$, or in other words that the theorem holds for $n + 1$.

By the principle of induction, the theorem holds for all $n \in \mathbb{N}$. $\square$

45. $\forall n \in \mathbb{N} \,.\, \sum_{x=n}^{2n} x = \frac{3(n+1)n}{2}$

Proof.

We proceed by induction.

Base Case When $n = 0$ we have $\sum_{x=0}^{0} 0 = 0$ and $\frac{3(0)9}{2} = 0$, so the theorem holds for $n = 0$.

Inductive step Assume the theorem holds for some $n \in \mathbb{N}$: that is, $\sum_{x=n}^{2n} x = \frac{3(n+1)n}{2}$. Consider the sum evaluated at $n+1$:

$$\begin{aligned}
\sum_{x=n+1}^{2(n+1)} x &= -n + 2n + 1 + 2n + 2 + \sum_{x=n}^{2n} x \\
&= 3n + 3 + \sum_{x=n}^{2n} x \\
&= 3n + 3 + \frac{3(n+1)n}{2} \\
&= 3n + 3 + \frac{3n^2 + 3n}{2} \\
&= \frac{6n + 6 + 3n^2 + 3n}{2} \\
&= \frac{3(n^2 + 3n + 2)}{2} \\
&= \frac{3(n+2)(n+1)}{2} \\
&= \frac{3\big((n+1)+1\big)(n+1)}{2}
\end{aligned}$$

which means the theorem holds at $n+1$ as well.

By the principle of induction, the theorem holds for all $n \in \mathbb{N}$. □

46. $\forall x \in \{a \mid a \in \mathbb{Z} \wedge a \geq -1\}$. $\sum_{k=-1}^{x} 12 - 2k = 26 + 11x - x^2$

Proof.

We proceed by induction.

Base Case When $x = -1$ we have $\sum_{k=-1}^{-1} 12 - 2k = 14 = 26 - 11 - 1$, so the theorem holds for $x = -1$.

Inductive step Assume the theorem holds for some x; that is, $\sum_{k=-1}^{x} 12 - 2k = 26 + 11x - x^2$. Consider the sum evaluated at $x + 1$:

$$\begin{aligned}
\sum_{k=-1}^{x+1} 12 - 2k &= 12 - 2(x+1) + \sum_{k=-1}^{x} 12 - 2k \\
&= 10 - 2x + 26 + 11x - x^2 \\
&= (11 - 1) - 2x + 26 + 11x - x^2 \\
&= 26 + (11 + 11x) - (1 + 2x + x^2) \\
&= 26 + 11(x+1) - (x+1)^2
\end{aligned}$$

which means the theorem holds at $x + 1$ as well.

By the principle of induction, the theorem holds for all $x \in \{a \mid a \in \mathbb{Z} \wedge a \geq -1\}$. $\square$

47. $\displaystyle\sum_{i=1}^{n} \frac{2}{3^i} = \frac{3^n - 1}{3^n}$

Proof.

We proceed by induction.

Base Case When $n = 1$ we have $\displaystyle\sum_{i=1}^{2} \frac{2}{3^i} = \frac{2}{3} = \frac{3^1 - 1}{3^i}$.

Inductive step Assume the theorem holds for some $k \in \mathbb{Z}^+$: that is, $\displaystyle\sum_{i=1}^{k} \frac{2}{3^i} = \frac{3^k - 1}{3^k}$. Adding $\frac{2}{3^{k+1}}$ to both sides we get $\displaystyle\frac{2}{3^{k+1}} + \sum_{i=1}^{k} \frac{2}{3^i} = \frac{3^k - 1}{3^k} + \frac{2}{3^{k+1}}$; the left-hand side is equivalent to $\sum_{i=1}^{k} \frac{2}{3^i}$ and the right-hand side can be re-written as

$$\begin{aligned}
& \frac{3^k - 1}{3^k} + \frac{2}{3^{k+1}} \\
= \quad & \frac{(3^k - 1) \cdot 3}{3^k \cdot 3} + \frac{2}{3^k \cdot 3} \\
= \quad & \frac{(3^{k+1} - 3) + 2}{3^{k+1}} \\
= \quad & \frac{3^{k+1} - 1}{3^{k+1}}
\end{aligned}$$

which is the theorem at $n = k + 1$.

By the principle of induction, the revised theorem holds for all $n \in \mathbb{Z}^+$. □

You might also try doing inductive proofs with other summation formulae, such as

$$
\begin{aligned}
\sum_{i=0}^{n} i^2 &= \frac{(n+1)(2n+1)(n)}{6} \\
\sum_{i=1}^{n+1} i^2 &= \frac{(n+2)(2n+3)(n+1)}{6} \\
\sum_{i=2}^{n+2} i^2 &= \frac{(n+3)(2n+5)(n+2)}{6} \\
6\sum_{i=0}^{n} i^3 - i &= \binom{n+2}{4} \\
\sum_{x=0}^{n} \frac{x^2-1}{x+1} &= \frac{(n+1)(n-1)}{2} \\
\sum_{x=0}^{n} x^3 - x^2 &= \frac{(n+1)(3n+2)(n)(n-1)}{12} \\
\sum_{i=0}^{n} 3i^2 + 2i &= \frac{(2n+3)(n+1)(n)}{2} \\
\sum_{x=n}^{n^2} x &= \frac{n+n^4}{2} \\
\sum_{x=0}^{2n} (-1)^x x &= n \\
\sum_{i=1}^{n} \frac{1}{2^i} &= \frac{2^n-1}{2^n} \\
\sum_{k=-n}^{0} k &= \frac{(n+1)n}{-2} \\
\sum_{i=1}^{n} \frac{1}{3^i} &= \frac{3^n-1}{3^n 2} \\
\forall k \neq 1 \;.\; \left(\sum_{i=1}^{n} \frac{1}{k^i}\right. &= \left.\frac{k^n-1}{k^n(k-1)}\right)
\end{aligned}
$$

Note: at least one of the above formulae is false. In the process of proving it you should find the normal methods not working, revealing the non-truth.