CS 2102 - DMT1 - Spring 2020 — Luther Tychonievich
Practice exercise in class friday april 17, 2020

Practice 11

problem 1 *Arithmetic*

1. Re-write $p^r = w$ without a exponent function: $\log_p(w) = r$ ______.

2. Rewrite to contain $\leq$ one log: $\dfrac{\log_3(7)}{\log_3(5)}$: $\log_5(7)$ ______.

3. Re-write $\log_2(16x^3)$ with no constants or operators in a log's argument: $4 + 3\log_2(x)$ ______.

4. What is $\log_3(5)\log_5(3)$? 1 ______.

5. Re-write $\log_3(x) = y$ without a log function: $3^y = x$ ______.

6. Re-write $\log_a(b)$ with a base-c log: $\dfrac{\log_c a}{\log_c b}$ ______.

7. Re-write $\log_2(a^2 \cdot b)$ with no constants or operators in a log's argument: $2\log_2(a) + \log_2(b)$ ______.

8. Suppose $\log_a(b) = \frac{2}{3}$. What is $\log_b(a)$? 1.5 ______.

PROBLEM 2 *Proof by Contradiction*

9. Complete the following proof that $\forall x \in \mathbb{Z}^+ \,.\, \big(\log_3(x) \in \mathbb{Q}^+\big) \to \big(\exists n \in \mathbb{N} \,.\, x = 3^n\big)$.

Proof. Assume that the implication does not hold; that is, that $\big(\log_3(x) \in \mathbb{Q}^+\big) \wedge \big(\nexists n \in \mathbb{N} \,.\, x = 3^n\big)$. Since $\log_3(x) \in \mathbb{Q}^+$, there are positive integers a and b such that $\log_3(x) = \dfrac{a}{b}$. Re-writing that equation,

$$\begin{aligned}
\log_3(x) &= \frac{a}{b} \\
b\log_3(x) &= a \\
\log_3(x^b) &= a \\
x^b &= 3^a
\end{aligned}$$

Since a and b are positive integers, both sides of the last equation above are integers. By the fundamental theorem of arithmetic, both sides must have the same prime factors, meaning that all of x's factors must be 3. But that contradicts our assumption that $\nexists n \in \mathbb{N} \,.\, x = 3^n$.

Because the assumption led to a contradiction, it must be false; thus,

$$\big(\log_3(x) \in \mathbb{Q}^+\big) \to \big(\exists n \in \mathbb{N} \,.\, x = 3^n\big)$$

□

10. Complete the following proof that $\log_2(3)$ is irrational.

Proof. Because $3 > 2$, $\log_2(3) > 1$. Assume that $\log_2(3)$ is rational. Then $\log_2(3) = \frac{a}{b}$, where a and b are positive integers. Re-writing this equation we get

$$\log_2(3) = \frac{a}{b}$$

$$\begin{aligned}
\log_2(3) &= \frac{a}{b} \\
b\log_2(3) &= a \\
\log_2(3^b) &= a \\
3^b &= 2^a
\end{aligned}$$

Since a and b are positive integers, both sides of the last equation above are integers. But they clearly share no prime factors, which contradicts the fundamental theorem of arithmetic.

Because the assumption led to a contradiction, it must be false and $\log_2(3)$ must be irrational. □

Want additional practice? Try the following:

Rewrite to contain $\leq$ one $\log$ without fractions inside the $\log$ (show your work)

- $\log_2(5) + \log_2(3)$
- $\log_5(24) - \log_5(4)$
- $\log_3(5) + \log_3(2)$
- $\log_3(5) + \log_9(0.2)$
- $\log_3\left(\frac{5}{27}\right)$
- $7^{\frac{\log_3(5)}{\log_3(7)}}$

Complete

- $\log_{\sqrt{3}}(5) = \log_3\left(\qquad\qquad\qquad\right)$
- $\log_a(b)\log_a\left(\qquad\qquad\qquad\right) = 1$
- $\log_a(b)\log_b\left(\qquad\qquad\qquad\right) = 1$
- $\log_3(13) = \log_3(5) + \log_3\left(\qquad\qquad\qquad\right)$
- $3^{\log_5(7)} = 7^{\log_{\square}(\square)}$
- $\log_4(9) = \log_2\left(\underline{\qquad\qquad\qquad}\right)$
- $\log_9(4) = \log_3\left(\underline{\qquad\qquad\qquad}\right)$

Prove that

- $\left(\log_a(b) = \log_b(a)\right) \rightarrow \left(a = b\right)$. Both direct proof and contradiction should be able to work here.
- $\log_5(7) \notin \mathbb{Q}$
- "$\forall n \in \left\{i \mid i \in \mathbb{Z} \land 1 < i < x\right\}\ .\ \log_i(x) \notin \mathbb{Q}$" is true for all prime numbers x. Use contradiction.
- $3\log_2(10) < 10$. Direct proof should be enough.
- $\log_3(10) > 2$. Direct proof should be enough.

Re-write

- $\log_3(x^q)$ without exponentiation
- $\log_{10}(x^7)$ without exponentiation
- $\log_4(x)$ using base-3 log(s) instead of base-4
- $\log_w(8)$ using base-x log(s) instead of base-w