PROBLEM 1 Counting

Number and write your answers to the following. You may use factorial, choose, and arithmetic notation, but may not use ellipses. For example, the following are all OK: \[120 \quad 5! \quad \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} \quad \left(\begin{array}{c} 5 \\ 3 \end{array}\right)\] however, the following is *not* OK: \[10 + 9 + \cdots + 2 + 1\].

There are 10 digits: 0 through 9.
There are 26 letters: 'a' through 'z'.

1. \[2^{10} = 1024\] How many sets of digits are there?

2. \[
\begin{align*}
\binom{10}{6} &= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 5040 \\
\end{align*}
\] How many 6-element sets of digits are there?

3. \[26 \cdot (10 + 26)^5 = 1572120576\] How many 6-character identifiers can be made with a leading letter following by 5 digits or letters (e.g. "cs2102" or "dmt1ok" but not "2012ok")?

4. \[
\frac{6}{6^3} = \frac{1}{6^2} = \frac{1}{36}
\] I roll three fair six-sided dice. What is the probability I roll the same number on all three?

5. \[
\frac{6!}{2!} - 1 = 359
\] Two strings are anagrams if they are distinct but can have their characters rearranged to be the same. How many anagrams of "cs2102" are there?
Problem 2  Proofs

Prove by induction that

\[ \forall n \in \mathbb{N} . \left( \sum_{x=0}^{n} (2x + 1) \right) = (n + 1)^2 \]

Proof. We proceed by induction.

Base Case  When \( n = 0 \) we have \( \sum_{x=0}^{0} (2x + 1) = 1 = (2 \cdot 0 + 1)^2 \), so the theorem holds for \( n = 0 \).

Inductive step  Assume the theorem holds for some \( n = k \); that is, \( \sum_{x=0}^{k} (2x + 1) = (k + 1)^2 \). Consider the sum evaluated at \( k + 1 \):

\[
\sum_{x=0}^{k+1} (2x + 1) = (2(k + 1) + 1) + \sum_{x=0}^{k} (2x + 1)
= (2k + 3) + (k + 1)^2
= 2k + 3 + k^2 + 2k + 1
= k^2 + 4k + 4
= (k + 2)^2
= ((k + 1) + 1)^2
\]

which means the theorem holds at \( k + 1 \) as well.

By the principle of induction, the theorem holds for all \( n \in \mathbb{N} \). \( \square \)