1. Re-write $p^r = w$ without a exponent function: $\log_p(w) = r$.

$\log\left(\frac{x}{y}\right) = \log x - \log y$

$\log_a(b) = \frac{\log_x(b)}{\log_x(a)}$

2. Re-write $\frac{\log_3(7)}{\log_3(5)}$ with only one log: $\log_5(7)$.

$\log_2(16) + \log_2(x^3)$ ($16 = 2^4$)

3. Re-write $\log_2(16x^3)$ with no constants or operators in a log's argument: $4 + 3\log_2(x)$ (lg).

$\log_5\left(3^{\log_3(5)}\right) = \log_5(5) = 1$

4. What is $\log_3(5)\log_5(3)$? 1.

$\frac{1}{\log_3(5)} = \log_5(3)$

5. Re-write $\log_3(x) = y$ without a log function: $x = 3^y$.

6. Re-write $\log_a(b)$ with a base-c log: $\frac{\log_c(b)}{\log_c(a)}$.

$\log_2(a^2) + \log_2(b)$

7. Re-write $\log_2(a^2 \cdot b)$ with no constants or operators in a log's argument: $2\lg(a) + \lg(b)$.

$\frac{\log_5(b)}{\log_5(a)} = 1$ $\quad \frac{1}{\log_9(9)} = \frac{2}{3}$

8. Suppose $\log_a(b) = \frac{2}{3}$. What is $\log_b(a)$? $\frac{3}{2}$.

9. Complete the following proof that $\forall x \in \mathbb{Z}^+ \,.\, (\log_3(x) \in \mathbb{Q}^+) \to (\exists n \in \mathbb{N} \,.\, x = 3^n)$.

Proof. Assume that the implication does not hold; that is, that $(\log_3(x) \in \mathbb{Q}^+) \wedge (\nexists n \in \mathbb{N} \,.\, x = 3^n)$. Since $\log_3(x) \in \mathbb{Q}^+$, there are positive integers a and b such that $\log_3(x) = \frac{a}{b}$. Re-writing that equation,

$\log_3(x) = \frac{a}{b}$

$b\log_3(x) = a$

$\log_3(x^b) = a$

$x^b = 3^a$

Since a and b are positive integers, both sides of the last equation above are integers. By the fundamental theorem of arithmetic, both sides must have the same prime factors, meaning that all of x's factors must be 3. But that contradicts our assumption that $\nexists n \in \mathbb{N} \,.\, x = 3^n$.

Because the assumption led to a contradiction, it must be false; thus,

$$(\log_3(x) \in \mathbb{Q}^+) \to (\exists n \in \mathbb{N} \,.\, x = 3^n)$$

□

10. Complete the following proof that $\log_2(3)$ is irrational.

Proof. Because $3 > 2$, $\log_2(3) > 1$. Assume that $\log_2(3)$ is rational. Then $\log_2(3) = \frac{a}{b}$, where a and b are positive integers. Re-writing this equation we get

$\log_2(3) = \frac{a}{b}$ (×b) → $b\log_2(3) = a$ → (idem) $\log_2(3^b) = a$ → (2^x) $2^{\log_2(3^b)} = 2^a$ → $3^b = 2^a$

$3 = 2^{a/b}$ → (2^x to both sides) → (x^b to both sides) $3^b = 2^a$

Since a and b are positive integers, both sides of the last equation above are integers. But they clearly share no prime factors, which contradicts the fundamental theorem of arithmetic.

Because the assumption led to a contradiction, it must be false and $\log_2(3)$ must be irrational. □

Write with at most one log **not containing internal operators (show your work)**

- $\log_2(5) + \log_2(3)$

  $\log_2(5 \cdot 3) = \log_2(15)$

- $\log_5(24) - \log_5(4)$

  $\log_5\left(\frac{24}{4}\right) = \log_5(6) = \log_5(3) + \log_5(2)$

- $\log_3(5) + \log_3(2)$

  $\log_3(10)$

- $\log_3(5) + \log_9(0.2)$

  $\frac{1}{2}\log_3\left(\frac{1}{5}\right) = \frac{\log_3\left(\frac{1}{5}\right)}{\log_3(9)}$ → $\log_3\left(5^{-1/2}\right) + \log_3(5^1)$ → $\log_3\left(5^{1/2}\right)$ → $\frac{1}{2}\log_3(5)$

  $\frac{\log_9(5)}{\log_9(3)} + \log_9\left(\frac{1}{5}\right) = 2\log_9(5) + \log_9\left(\frac{1}{5}\right)$ → $\log_9(5^2) + \log_9(5^{-1})$ → $\log_9(5)$

  $\log_9(3) = \log_9(9^{1/2}) = \log_9(1)\frac{1}{2} = \frac{1}{2}$

- $\log_3\left(\frac{5}{27}\right)$

  $\log_3(5) - \log_3(27) = \log_3(5) - 3$ ($27 = 3^3$)

- $7^{\frac{\log_3(5)}{\log_3(7)}}$

  $= 7^{\log_7(5)} = 5$

Complete

- $\log_{\sqrt{3}}(5) = \log_3(25)$

  $\frac{\log_3(5)}{\log_3(\sqrt{3})} = \frac{\log_3(5)}{\frac{1}{2}} = 2\log_3(5) = \log_3(5^2) = \log_3(25)$

- $\log_a(b)\log_a\left(a^{\log_a(b)^{-1}}\right) = 1$

  $\log_a\left(b^{\log_a(x)}\right) = 1$ → $\log_a\left(b^{\log_b(x)/\log_b(a)}\right) = 1$ → $\log_a\left(b^{\log_b(x)\cdot 1/\log_b(a)}\right) = 1$ → $\log_a\left(x \cdot b^{1/\log_b(a)}\right) = 1$ → $x \cdot b^{1/\log_b(a)} = a$ → $x = \frac{a}{b^{1/\log_b(a)}}$

  $x^{y+z} = x^y x^z$

- $\log_a(b)\log_b(a) = 1$

  $\log_a(b) = \frac{1}{\log_b(a)}$

- $\log_3(13) = \log_3(5) + \log_3\left(\frac{13}{5}\right)$

  $\log_3(5 \cdot x)$

- $3^{\log_5(7)} = 7^{\log_7(5)}$

  $7^{\log_7\left(3^{\log_5(7)}\right)}$ → $\log_7 3^{\log_5(7)} = \log_x(y)$ → $\log_5(7)\log_7(3) = \log_7(y)$ → $\frac{\log_7(3)}{\log_7(5)} = \log_5(3) = \log_x(y)$

- $\log_4(9) = \log_2(3)$

  $\frac{\log_2(4)}{2} = \frac{\log_2(4)}{\log_2(4)}$ → $\frac{1}{2}\log_2(4) = \log_2(4^{1/2})$

- $\log_9(4) = \log_3(2)$

  $\frac{\log_3(4)}{\log_3(4)} = 2$

Prove that

- $(\log_a(b) = \log_b(a)) \to (a = b)$. Both direct proof and contradiction should be able to work here.

- $\log_5(7) \notin \mathbb{Q}$

- "$\forall n \in \{i \mid i \in \mathbb{Z} \wedge 1 < i < x\} \,.\, \log_i(x) \notin \mathbb{Q}$" is true for all prime numbers x. Use contradiction.

- $3\log_2(10) < 10$. Direct proof should be enough.

- $\log_3(10) > 2$. Direct proof should be enough.

Re-write

- $\log_3(x^q)$ without exponentiation

- $\log_{10}(x^7)$ without exponentiation

- $\log_4(x)$ using base-3 log(s) instead of base-4

- $\log_w(8)$ using base-x log(s) instead of base-w