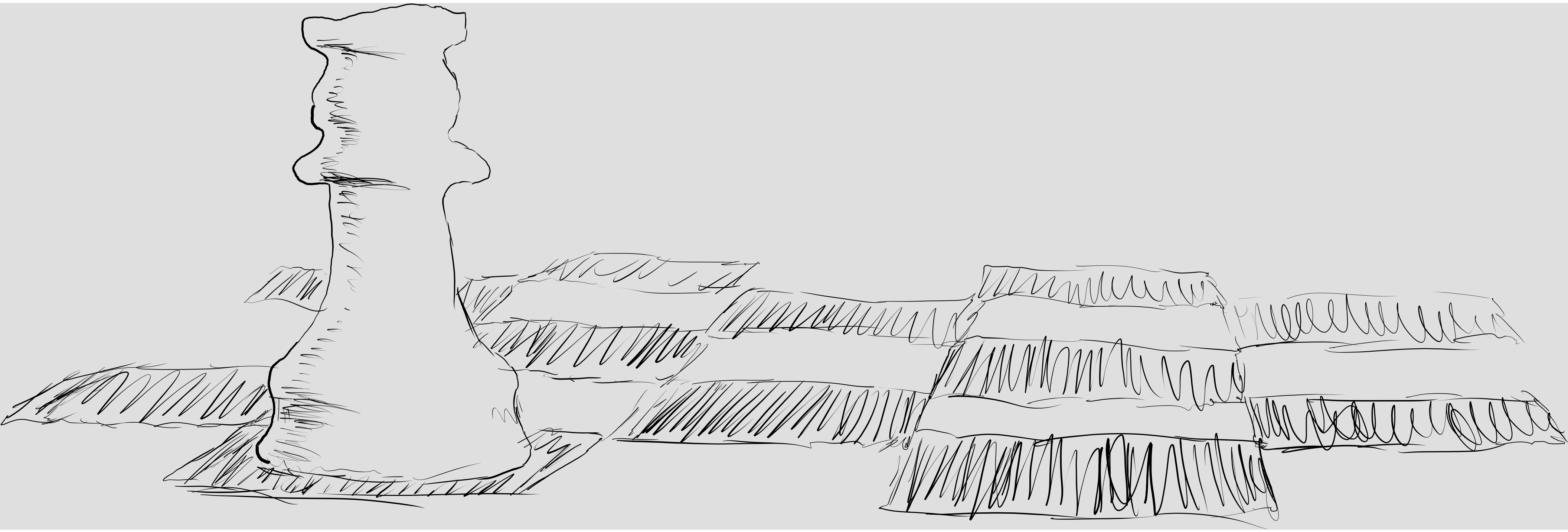


$(x-1, y+1)$ $(x+1, y+1)$

(x, y)

$(x-1, y-1)$ $(x+1, y-1)$

$x-3, y-3$

Invariant

"color of square"

$P(x,y)$: $x+y$ is odd $\quad (x+y) \bmod 2 \quad$ parity$(x+y)$

Pf: - invariant is invariant

- start & end have different

$$(x, y) \to (x \pm a, y \pm a) \ \forall a > 0$$

Thm: no set of moves get from (x, y) to $(x+1, y)$

define $\text{parity}(x+y) = (x+y) \bmod 2$

lemma: $\forall$ transitions, Parity unchanged

pf transition is $(x,y) \to (x \pm a, y \pm a)$

case $(x+a, y+a)$ $\to (x+y+\underbrace{2a}_{\text{even}}) \bmod 2 = (x+y) \bmod 2$

case $(x+a, y-a)$ $\to (x+y) \bmod 2 = (x+y) \bmod 2$

Thm: from (x,y), $(x, y+1)$ is unreachable

assum is reachable.

sub-proof by induction

then, by lemma, (x,y) and $(x, y+1)$ have $\boxed{\text{same parity}}$

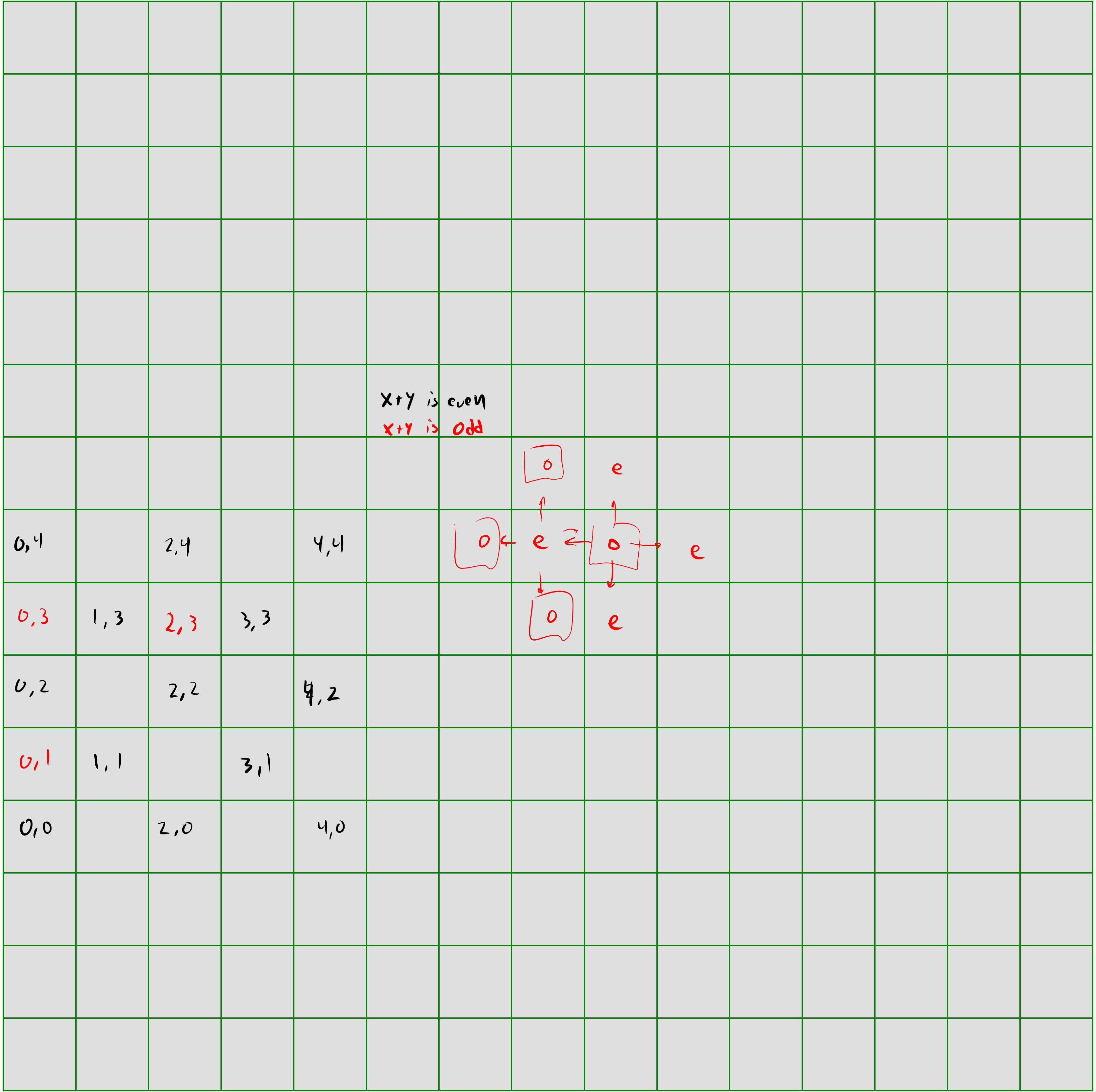
base case: no transition. trivially (x,y) and (x,y) have same parity

induction: assum k transition does not change parity. Then $k+1$ does not

take last transition by lemma

but $\boxed{\underbrace{(x+y) \bmod 2}_{z} \neq \underbrace{(x+y+1) \bmod 2}_{1-z}}$ $\perp$

Contradiction



x+y is even
x+y is odd
o
e
o
e
o
e
o
e
0,4
2,4
4,4
0,3
1,3
2,3
3,3
0,2
2,2
4,2
0,1
1,1
3,1
0,0
2,0
4,0

```
gcd(a,b)
    if b < a, swap a & b
    while a ≠ 0
        b -= a
        if b < a, swap a & b
    return b
```

Progress: transition makes a, b smaller

1. smaller num

$$(a, b) \to \begin{cases} (a,\ b-a) \\ \text{xor} \\ (b-a,\ a) \end{cases}$$

invariant: $\gcd(\overset{3}{a}, \overset{3}{b}) = \gcd(\overset{3}{a}, \overset{0}{b-a})$

case $a = b$, return a ✓

case $a \neq b$ $\quad \gcd(a,b) = x$

$$x\frac{a}{x} \neq x\frac{b}{x} \qquad \overbrace{x\frac{b}{x} - x\frac{a}{x}}^{\text{int}}$$

$$x\Big(\underbrace{\frac{b}{x} - \frac{a}{x}}_{\text{int}}\Big)$$

2. Same gcd

Sort a list

invariant

list length
elements it contains

Progess function

find an out-of-order pair & swap them.

| 3 | 1 | 4 | 1 | 5 | 9 |

3

3 1 1 4 5 9

2

1 3 1 4 5 9

1

1 1 3 4 5 9