CS3102 Theory of Computation
Problem Set 1
Department of Computer Science, University of Virginia

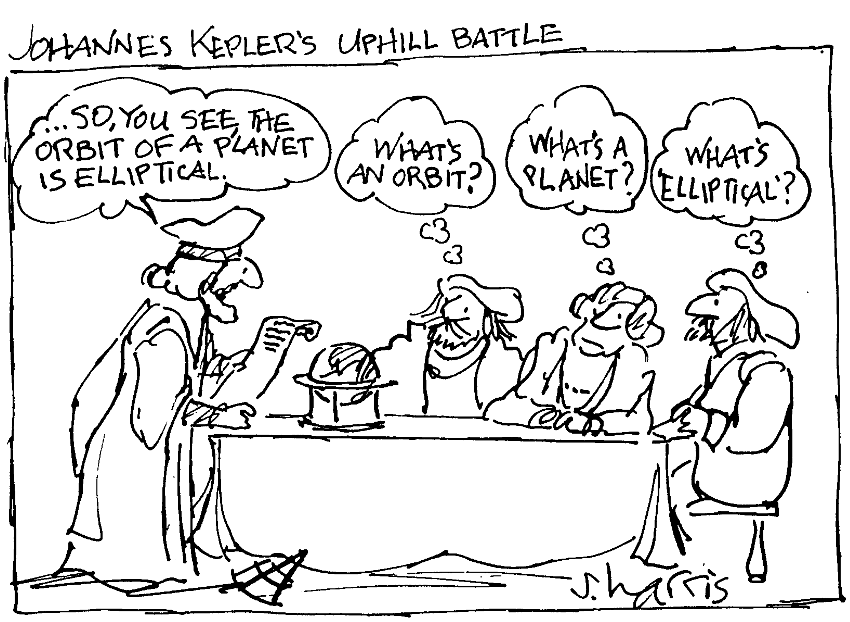
Gabriel Robins

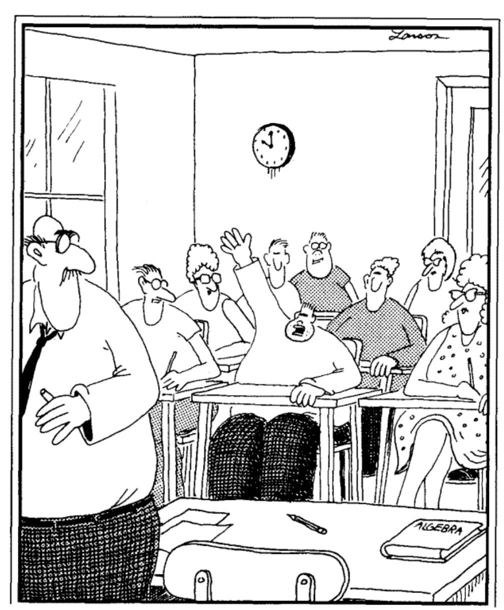
Please start solving these problems immediately, don't procrastinate, and work in study groups. Please prove all your answers; informal arguments are acceptable, but please make them precise / detailed / convincing enough so that they can be easily made rigorous if necessary. To review notation and definitions, please read the "Basic Concepts" summary posted on the class Web site, and also read the corresponding chapters from the Sipser textbook and Polya's "How to Solve It".

Please do not simply copy answers that you do not fully understand; on homeworks and on exams we reserve the right to ask you to explain any of your answers verbally in person (and we have exercised this option in the past). Please familiarize yourself with the UVa Honor Code as well as with the course Cheating Policy summarized on page 3 of the Course Syllabus. To fully understand and master the material of this course typically requires an average effort of at least six to ten hours per week, as well as regular meetings with the TAs and attendance of the weekly problem-solving sessions.

This is not a "due homework", but rather a "pool of problems" meant to calibrate the scope and depth of the knowledge / skills in CS theory that you (eventually) need to have for the course exams, becoming a better problem-solver, be able to think more abstractly, and growing into a more effective computer scientist. You don't necessarily have to completely solve every last question in this problem set (although it would be great if you did!). Rather, please solve as many of these problems as you can, and use this problem set as a resource to improve your problem-solving skills, hone your abstract thinking, and to find out what topics you need to further focus on and learn more deeply. Recall that most of the midterm and final exam questions in this course will come from these problem sets, so your best strategy of studying for the exams in this course is to solve (including in study groups) as many of these problems as possible, and the sooner the better!

Advice: Please try to solve the easier problems first (where the meta-problem here is to figure out which are the easier ones ☺). Don't spend too long on any single problem without also attempting (in parallel) to solve other problems as well. This way, solutions to the easier problems (at least easier for you) will reveal themselves much sooner (think about this as a "hedging strategy" or "dovetailing strategy").

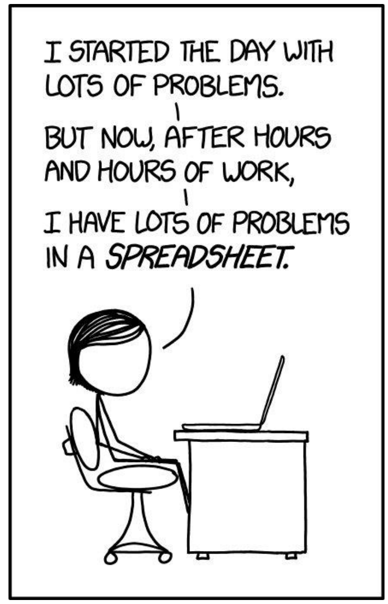




"Mr. Osborne, may I be excused? My brain is full."

1. Solve the following problems from [Sipser, Second Edition]. Pages 26-27: 0.3, 0.4, 0.5, 0.7, 0.8, 0.10, 0.11, 0.12.

2. True or false:
   a. $\emptyset \subseteq \emptyset$
   b. $\emptyset \subset \emptyset$
   c. $\emptyset \in \emptyset$
   d. $\{1,2\} \in 2^{\{1,2\}}$
   e. $\{1,2\} \subseteq 2^{\{1,2\}}$
   f. $\{x,y\} \in \{\{x,y\}\}$



3. Write the following set explicitly: $2^{\{1,\,2\}} \times \{v,w\}$

4. Prove without using induction that for an arbitrary finite set S, the sets 2^S and $\{0,1\}^{|S|}$ have the same number of elements.

5. Which of the following sets are closed under the specified operations?
   a) $\{x \mid x \text{ is an odd integer}\}$, multiplication
   b) $\{y \mid y=2n, n \text{ some integer}\}$, subtraction
   c) $\{2m+1 \mid m \text{ some integer}\}$, division
   d) $\{z \mid z=a+bi \text{ where a and b are real, } |a||b| > 0, \text{ and } i=\sqrt{-1}\}$, exponentiation

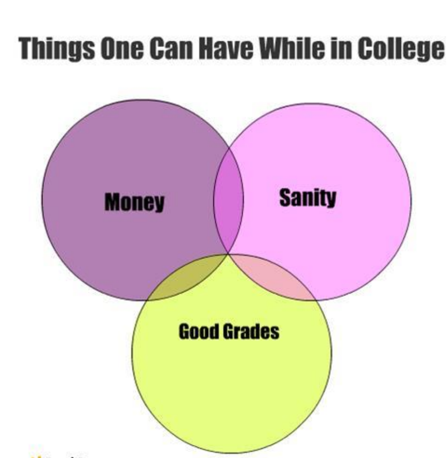
6. Is the transitive closure of a symmetric closure of a binary relation necessarily reflexive? (Assume that every element of the "universe" set participates in at lease one relation pair.)

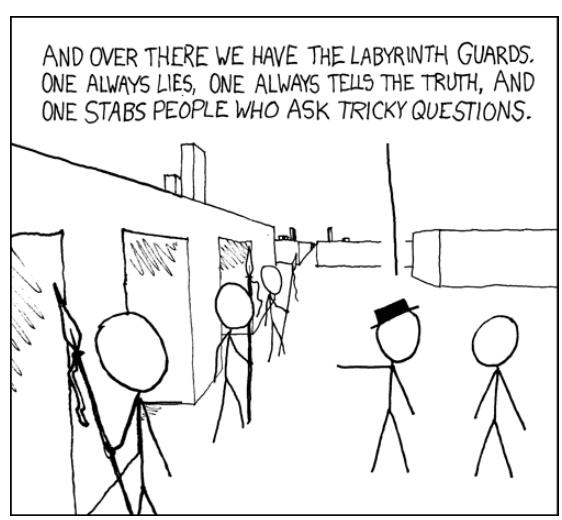
7. True or false: a countable union of countable sets is countable.

8. True or false: if T is countable, then the set $\{S \mid S \subseteq T, S \text{ finite}\}$ is also countable.

9. Give a simple bijection for each one of the following pairs of sets:
   a) the integers, and the odd integers.
   b) the integers, and the positive integers.
   c) the naturals, and the rationals crossed with the integers.

10. Is there a bijection between the closed unit interval $\{x \mid x \in \mathbf{R}, 0 \le x \le 1\}$ and $\mathbf{R}$?





11. Generalize $|S| < |2^S|$ to arbitrary infinite sets (not necessarily countable ones).

12. What is the cardinality of each of the following sets ?
   a. The set of all polynomials with rational coefficients.
   b. The set of all functions mapping reals to reals.
   c. The set of all possible Java programs.
   d. The set of all finite strings over the alphabet {0,1,2}.
   e. The set of all 5x5 matrices over the rationals.
   f. The set of all points in 3-dimensional Euclidean space.
   g. The set of all valid English words.
   h. $\{\emptyset, \mathbf{N}, \mathbf{Q}, \mathbf{R}\}$
   i. $\mathbf{N} \times \mathbf{Z} \times \mathbf{Q}$
   j. $\mathbf{R} - \mathbf{Q}$

13. Prove without using induction that n^4-4n^2 is divisible by 3 for all $n \geq 0$.

14. How many distinct boolean functions on N variables are there? In other words, what is the cardinality of $|\{f \mid f:\{0,1\}^N \rightarrow \{0,1\}\}|$?

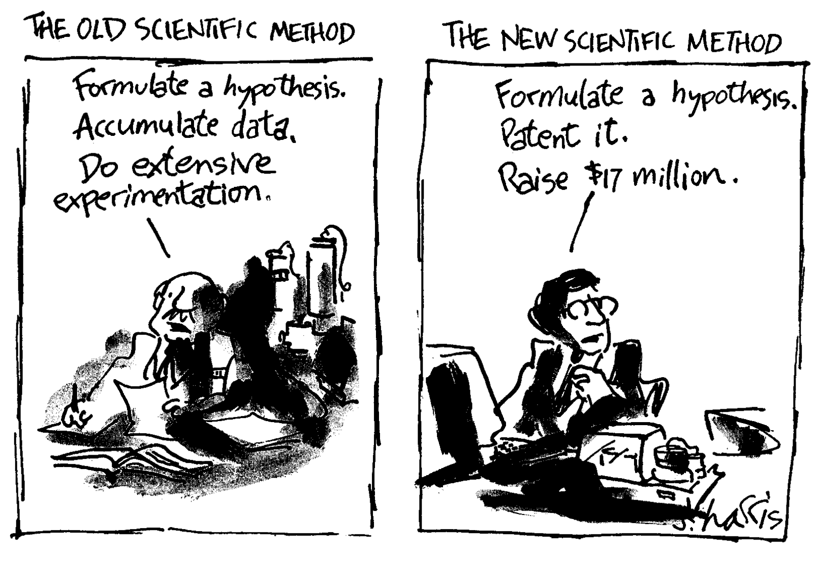
15. How many distinct N-ary functions are there from finite set A to finite set B? Does this generalize the previous question?

16. Show that in any group of people, there are at least two people with the same number of acquaintances within the group. Assume the "acquaintance" relation is symmetric but non-reflexive.

17. Show that in any group of six people, there are either 3 mutual strangers or 3 mutual acquaintances.

18. A clique in a graph is a complete subgraph (i.e., all nodes are connected with edges). Show that every graph with N nodes contains a clique or the complement of a clique, of size at least ½ $\log_2 N$.

"The periodic table."



19. Show that the set difference of an uncountable set and a countable set is uncountable.

20. Show that the intersection of two uncountable sets can be empty, finite, countably infinite, or uncountably infinite.

21. For an arbitrary language L, prove or disprove each of the following:

a) $(L^*)^*=L^*$

b) $L^+=L^*-\{\wedge\}$

22. Characterize completely the cardinalities of <u>all</u> sets of identical test tubes that can be spun simultaneously in a 360-hole centrifuge in a balanced way (i.e. 1 test tube cannot be spun, but 2, 3, 4, and 5 can, etc.)

23. Prove that there are an infinity of prime numbers.

24. Prove or disprove: for any arbitrarily large natural number N, there exists N consecutive composite natural numbers (i.e. argue whether there exists "prime deserts" of arbitrarily large sizes).

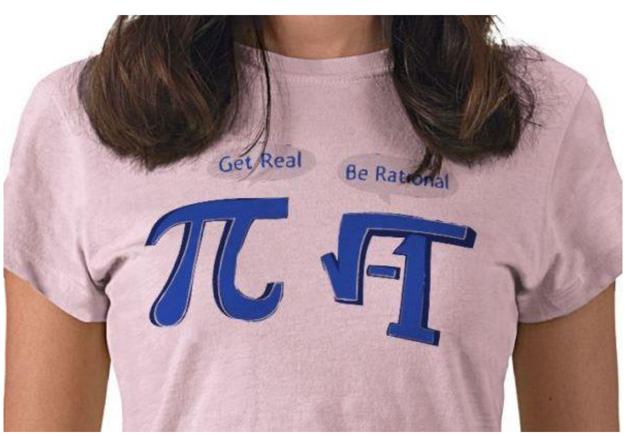
25. Compute the infinite sum $(1/16) + (1/16)^2 + (1/16)^3 + (1/16)^4 + \ldots =$? without using induction.

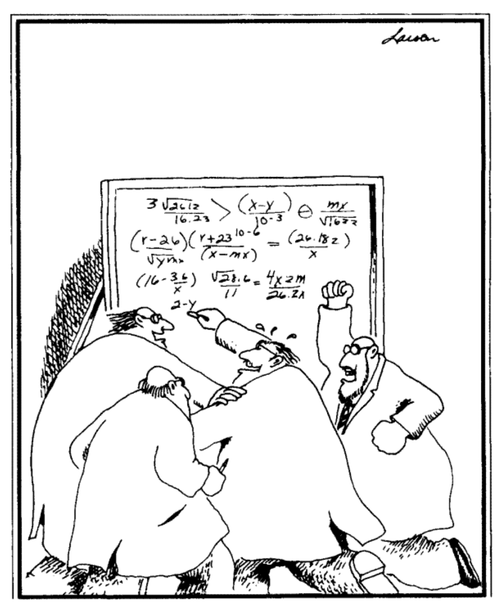
26. Find a formula (as a function of n) for $1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 =$? and prove it using a picture (and without using induction).

27. Prove that the square root of 2 is irrational.

28. Are the complex numbers closed under exponentiation? And if so, what is the value of i^i?

29. Does exponentiation preserve rationality? Does exponentiation preserve irrationality? i.e., are there two irrational numbers x and y such that x^y is rational?





"Go for it, Sidney! You've got it! You've got it! Good hands! Don't choke!"