Financial Mathematics: Homework #4

Ana Nora Evans

September 12, 2011

Problem 1

Alice takes a 10% simple interest loan from Bob with a one year term. After n days Bob has some financial trouble and he sells the loan to Katie. Since Katie is a nice and generous person she is not trying to take advantage of Bob's difficult situation and she asks for the same interest rate of 10%. How does the amount Bob gets from Katie compare with amount Bob would make if he would give Alice a 10% simple interest loan with term n days?
Your answer should be in terms of n, where $1 \leq n \leq 364$.

Answer:

Let P denote the money Bob lends to Alice.

If Bob gives Alice a 10% simple interest loan with term n days, then on the n^{th} day Bob receives from Alice $S_A = P(1+0.1\frac{n}{365})$.

To calculate how much Katie pays Bob for the loan, we need to determine, first, the maturity value, S, of the loan.

$$S = P(1+0.1 \cdot 1) = 1.1P$$

Thus, Katie pays Bob

$$S_K = \frac{1.1P}{1+0.1(1-\frac{n}{365})}$$

Since P appears as a factor in both S_A and S_K, the relationship between them does not depend on P. We can take $P = \$1$.

Depending on your level of comfort with inequalities there are different approaches. One approach is to solve the inequality

$$1+0.1\frac{n}{365} \geq \frac{1.1}{1+0.1(1-\frac{n}{365})}$$

Since this is a rather lengthy and not particularly interesting calculation, let's type

$$1+0.1*x/365 > 1.1/(1+0.1*(1-x/365)$$

in Wolfram Alpha. You will see that if $1 \le x \le 364$ then x is a solution. This means that the inequality is true when $1 \le x \le 364$.

Another approach is to execute

$$\text{plot } 1+0.1*x/365, 1.1/(1+0.1*(1-x/365)), \text{ from } x=1 \text{ to } 364$$

in Wolfram Alpha and notice that the graph of $1.1/(1+0.1*(1-x/365))$ is below the graph of $1+0.1*x/365$. Thus, $S_A \ge S_K$.

Problem 2

Modular arithmetic is a fancy name for the type of addition and subtraction you do when you calculate time.
Please read from http://mathdude.quickanddirtytips.com/what-is-modular-arithmetic.aspx about it (or any other resource).
a) Use modular arithmetic to calculate the term, in days, of a loan taken out on 5 March 2011 with a due date 22 January 2012?
b) Use modular arithmetic to calculate the term, in months, of a loan taken out on 2 December 2011 with a due date 2 March 2012?

Answer:

a) The number of days from 5 March 2011 to 22 January 2012 is

$$22-64 \; mod \; 365 = 363$$

b) The number of months from 2 December 2011 to 2 March 2012

$$3-12 \; mod \; 12 = 3$$

Note: This is, probably, not a good question, but I hope some of you enjoyed learning about this type of math. I thought it was easier because you don't have to deal with two cases: one when the dates are in the same year and one when the dates are in different years. The same formula works in both cases.

Problem 3 (Exercise 14, page 29)

A&R Plumbing contracts for a new home plumbing job. The contract stipulates that the home-owner will pay A&R $\$15,400$ in six months if the job is complete. Supplies are estimated to cost A&R $\$11,200$ now. In three months A&R also spends $\$2,000$. Assuming the job is done on time, find the NPV at 40% and the IRR.

Answer:

To determine $NPV_{@40\%}$ we bring all the money at the present using the formula $P = \frac{S}{1+it}$.

$$NPV_{@40\%} = -\$11,200 + \frac{-\$2,000}{1+0.4\cdot\frac{3}{12}} + \frac{\$15,400}{1+0.4\cdot\frac{6}{12}}$$

Since IRR is the interest rate such that $NPV_{@IRR} = 0$, then we need to solve the equation:

$$0 = NPV_{@IRR} = -\$11,200 + \frac{-\$2,000}{1 + IRR\frac{3}{12}} + \frac{\$15,400}{1 + IRR\frac{6}{12}}$$

To find the solution of this equation we type

$$0 = -11200 - 2000/(1 + x/4) + 15400/(1 + x/2)$$

in Wolfram Alpha and we choose the positive solution of the equation:

$$IRR = 36.31\%$$

Another method to solve the equation is to use the formula for the roots of a quadratic equation.

Problem 4 (Exercise 12, page 32)

Reliance Hardware owes their paint supplier $\$12,800$ on April 1. The supplier will agree to three payments of $\$4,000$ on June 1, $\$3,500$ on August 1, $\$2,500$ on October 1, provided that the balance is paid on December 1 and Reliance Hardware pays 8% interest. Find the final payment using the Merchant's Rule.

Answer:

The Merchant's Rule of calculating the balance at the due date is to bring all the money at the due date and add them, considering the loan negative and the payments positive. All the monies are moved forward using the formula $S = P(1 + it)$.

$$\begin{aligned} Balance = -\$12,800(1 + 0.08 \cdot \frac{8}{12}) + \$4,000(1 + 0.08 \cdot \frac{6}{12}) + \\ \$3,500(1 + 0.08 \cdot \frac{4}{12}) + \$2,500(1 + 0.08 \cdot \frac{2}{12}) \end{aligned}$$

Problem 5 (Exercise 14, page 33)

An $\$80,000$ home loan financed for 30 years at 7% has a monthly payment of $\$532.24$. Use the US Rule to find how much of the first two payments goes to the interest and how much goes to the principal.

Answer:

The balance at the time of the first payment is the sum beginning balance, $-\$80,000$ brought at the time of the first payment and the first payment:

$$B_1 = -\$80,000(1 + 0.07 \cdot \frac{1}{12}) + \$532.24 = -\$79,934.43$$

The balance at the time of the second payment is the sum of the balance at the first payment brought at the time of the second payment and the second payment:

$$B_2 = -\$79934.43(1+0.07\cdot\frac{1}{12}) + \$532.24 = -\$79,868.47$$

The part of the principal paid after the second payment is $\$80,000 - \$79,868.47 = \$131.53$. The part of the first two payments that goes to the interest is $2\cdot\$532.24 - \$131.53 = \$932.95$.

Problem 6 (Exercise 18, page 45)

Junior is planning to build a workshop. The $15,000 cost of the building will be financed with a bank discount loan at a rate of 8.2% for 2 years. What will be the amount financed?

Answer:

Junior needs $15,000 now. With a discount loan, the interest is paid upfront. The proceeds are $P = \$15,000$, the term is $t = 2$ and the discount rate is $d = 8.2\%$. To determine the amount of the loan we use the discount amount formula $S = \frac{P}{1-dt}$.

$$S = \frac{\$15,000}{1-0.082\cdot 2}$$

Note If Junior's workshop is not used for business, but only for hobbies, he should save the money first, and then build the workshop. He should not get in debt at a discount rate of 8.2%.

Problem 7 (Exercise 16, page 47)

A $50,000 financial instrument is worth $50,400 in 3 months. Find the rate of return at simple interest rate and at bank discount rate.

Answer:

This problem is asking you to calculate the interest rate and the discount rate of an investment of $50,000 that is worth $50,400 in 3 months.

To calculate the rate of return at simple interest we use the formula $I = S - P = Pit$. We are given the principal $P = \$50,000$, the maturity value $S = \$50,400$, and the term $t = 3$ months. We calculate i:

$$i = \frac{S-P}{Pt} = \frac{\$50,400 - \$50,000}{\$50,000\cdot\frac{3}{12}}$$

To calculate the bank discount rate we use the formula $D = S - P = Sdt$. We are given the proceeds $P = \$50,000$, the amount $S = \$50,400$, and the term $t = 3$ months. We calculate d:

$$d = \frac{S-P}{St} = \frac{\$50,400 - \$50,000}{\$50,400\cdot\frac{3}{12}}$$

Problem 8 (Exercise 18, page 47)

A woman invests $\$104,000$ on 8/2/03 and gets a return of $\$130,000$ on 6/20/04. Find the rate of return on a simple interest and bank discount basis.

Answer:

The term is $t = 365 - 214 + 171 = 322$ days.
To calculate the rate of return at simple interest we use the formula $I = S - P = Pit$.
We are given the principal, $P = \$104,000$, the maturity value, $S = \$130,000$, and the term, $t = 322$ days. We calculate i:

$$i = \frac{S-P}{Pt} = \frac{\$130,000 - \$104,000}{\$104,000 \cdot \frac{322}{365}}$$

To calculate the bank discount rate we use the formula $D = S - P = Sdt$. We are given the proceeds, $P = \$104,000$, the amount, $S = \$130,000$, and the term $t = 322$ days. We calculate d:

$$d = \frac{S-P}{St} = \frac{\$130,000 - \$104,000}{\$130,000 \cdot \frac{322}{365}}$$

Problem (Bonus)

In the lecture one notes I presented an example of interest calculation using the average daily balance and daily accrual. I obtained an interest of \$33.94 using average daily balance and \$34.09 using daily accrual.
Next, I will present a proof claiming that the interest calculated using average daily balance and the interest calculated using daily accrual are equal.
Let n be the number of days in the cycle. Let B_k be the daily balance of the credit card for the k^{th} day of the cycle.
First, I will calculate the interest using the average daily balance, denoted by I_{ADB}. The average daily balance is the sum of the balance of each day, divided by the number of days in the cycle, that is $\frac{1}{n}\sum_{k=1}^{n} B_k$. Thus

$$I_{ADB} = \left(\frac{1}{n}\sum_{k=1}^{n} B_k\right)\frac{APR}{365}n = \left(\sum_{k=1}^{n} B_k\right)\frac{APR}{365}$$

Second, I will calculate the interest using the daily accrual method, denoted by I_{DA}. The interest for the k^{th} day, denoted by I_k, is $I_k = B_k\frac{APR}{365}$. The interest using daily accrual is the sum of daily accrual.

$$I_{DA} = \sum_{k=1}^{n} I_k = \sum_{k=1}^{n}\left(B_k\frac{APR}{365}\right)$$

Since the real numbers are distributive, then $\sum_{k=1}^{n}\left(B_k\frac{APR}{365}\right) = \left(\sum_{k=1}^{n} B_k\right)\frac{APR}{365}$.
Thus, $I_{ADB} = I_{DA}$ In words, the interest calculated using average daily balance is mathematically

equal to the interest calculated daily accrual, when applied to the same data.
This seems to contradict the example from lecture 1.
Did I make a mistake in the calculation? Did I make a mistake in the proof? If both the calculations from the notes and the proof are correct, how do you explain the seemingly contradictory results?

Answer:

> There might be some mistakes in my calculations, but that is not the issue. The proof is correct. Rounding of decimal numbers is the cause of different results when using the two methods. In general, in the various steps in calculation, the result is a decimal number with an infinite number of non-zero digits after the decimal point. Since all the calculators work with only a finite number of digits after the decimal point, then they will truncate the numbers to a certain number of decimals. Mathematically, the two methods are the same. In practice, they can give different results. The difference between them is not big and depends on the rounding conventions made (e.g how many digits after the decimal point to consider). In my calculations I used two digits at after the decimal point and, probably, did not pay much attention to rounding up.