Financial Mathematics: Homework #6

Ana Nora Evans

October 12, 2011

Problem 1

Solve the inequality $3x+9<2x+1$.

Answer:

We learned a few inequality properties:
(P1) If $a<b$, then $a+c<b+c$. In other words, we can add or subtract from both sides and the inequality will keep the inequality sign ($<$ or $>$).
(P2) If $a<b$ and $c<0$, then $ac>bc$. In other words, we can multiply the inequality by a negative number, but we have to change the sign ($<$ is replaced with $>$, and $>$ is replaced by $<$).
(P2) If $a<b$ and $c>0$, then $ac<bc$. In other words, we can multiply the inequality by a positive number, and the inequality will keep the inequality sign ($<$ or $>$).

We will use the properties above to bring the inequality to a form $x<?$. First we use property P1 with $a=3x+9$, $b=2x+1$ and $c=-9$ (or subtract 9 from both sides):

$$\begin{aligned} 3x+9-9 &< 2x+1-9 \\ 3x &< 2x-8 \end{aligned}$$

We have x on both sides. We can use property P1 again, subtract $2x$ from both sides. It does not matter we don't know the value of x, the property P1 is true for all numbers.

$$\begin{aligned} 3x-2x &< 2x-8-2x \\ x &< -8 \end{aligned}$$

The solution to the inequality is $x<-8$, which means if we replace x by *any* number smaller than -8 the inequality will be true.

Problem 2

a) Prove that $x^2 \geq 0$ for all numbers x.
Hint: Consider two cases: $x \geq 0$ and $x < 0$.
b) Explain why we can't take the square root of a negative numbers.

Answer:

Part a:

If we multiply two positive numbers we get a positive number, and if we multiply two negative numbers we get a positive number. We conclude that $x^2 \geq 0$.

Another solution is to apply property P2 when $x < 0$. Multiplying by x, we get $xx > x \cdot 0 = 0$. If $x > 0$ we apply property P3 and we multiply by x to get $xx > 0x = 0$. The equality is the case $x = 0$. Therefore $x^2 \geq 0$ for all numbers x.

Part b:

The square root of a number x is a number y such that $y^2 = x$. By part a, for any number y, $y^2 \geq 0$. We conclude there is no such number y is x is negative.

Problem 3

Prove $(x^n)^m = x^{nm}$ for all numbers x and all natural numbers n and m.
Observation: Any number raised to the power zero is equal to 1. In mathematical notation, we write $x^0 = 1$ for all numbers x.

Answer:

$$\begin{aligned}
(x^n)^m &= \underbrace{x^n \cdot x^n \cdot \cdots \cdot x^n}_{m \text{ times}} \\
&= \underbrace{\underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ times}} \cdot \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ times}} \cdot \cdots \cdot \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ times}}}_{m \text{ times}} \\
&= \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \cdot m \text{ times}} \\
&= x^{nm}
\end{aligned}$$

Problem 4, Page 72, exercise 14

In this and all the subsequent chapters 5% means 5% compunded ______ or 5%(_).

Answer:

In this and all the subsequent chapters 5% means 5% compunded yearly or 5%(1).

Problem 4, Page 72, exercise 20

Toni invests $\$20,000$ for 3 years at 10%. Find the amount if:
a) Money is compounded quarterly.
b) Money is compounded monthly.
c) Money is compounded daily (use 360 days/year).

Answer:

In this problem 10% is the nominal interest rate. The interest rate per period is calculated as the nominal interest rate divided by the number of conversion periods per year, m, that is $\frac{10\%(m)}{m}$. The total number of conversion period, n, in a 3 year term is $3m$.

To determine the amount we use compound amount formula $S = P(1+i(m)/m)^n$.

a) If money is compounded quarterly, then $m = 4$. Thus $n = 12$, $i(m)/m = 10\%/4 = 0.1/4 = 0.025$, and

$$S = \$20,000(1+0.025)^{12} = \$26,897.78$$

b) If money is compounded monthly, then $m = 12$. Thus $n = 3 \cdot 12 = 36$, $i(m)/m = 10\%/12 = 0.1/12$, and

$$S = \$20,000(1+0.1/12)^{36} = \$26,963.64$$

b) If money is compounded daily, then $m = 360$. Thus $n = 3 \cdot 360 = 1080$, $i(m)/m = 10\%/360 = 0.1/360$, and

$$S = \$20,000(1+0.1/360)^{1080} = \$26,996.05$$

Problem 4, Page 76, exercise 1

Solve $S = P(1+i)^n$ for P and derive the formula for present value under compound interest.

Answer:

We divide both sides by $(1+i)^n$:

$$P = \frac{S}{(1+i)^n}$$

Since $\frac{1}{(1+i)^n} = (1+i)^{-n}$, then

$$P = S(1+i)^{-n}$$

Problem 4, Page 87, exercise 1

An effective rate problem is finding ______.

Answer:

An effective rate problem is finding the annual interest rate of an equivalent loan compounded yearly.

Problem 4, Page 87, exercise 10

Find the effective rate for a nominal interest rate of 5% if money is compounded
a) semiannually
b) quarterly
c) monthly
d) daily

Answer:

To determine the effective interest rate we consider two loans with the same principal P, the same amount P, the same term of one year, but different interest rates. One loan has the given interest rate $i(m)$. The other loan has the unknown interest rate $x(1)$. The amount for the first loan is given by the formula $S = P(1+i(m)/m)^m$. Here the number of total conversion periods is m since the term is one year. Similarly, the amount for the second loan is $S = P(1+x(1)/1)^1 = P(1+x)$. Since the amounts are the same for the two loans are the same, we have

$$P(1+i(m)/m)^m = P(1+x)$$

We divide both sides by P and we obtain

$$(1+i(m)/m)^m = 1+x$$

To determine x, we subtract 1 from both sides

$$x = (1+i(m)/m)^m - 1$$

In all parts, $i = 5\% = 0.05$.

Part a:

In this case $m = 2$. We apply the formula above

$$x = (1+0.05/2)^2 - 1 = 0.050625 = 5.0625\%$$

The effective interest rate of a loan with nominal interest rate 5% compounded semianually is 5.0625%.

Problem 4, Page 87, exercise 12

Which is higher: 7.2%(2) or 7%(12)?

Answer:

We can not compare the nominal interest rates directly. We have to calculate the effective interest rates for both loans.

The effective interest rate of 7.2%(2) is

$$i_1 = (1 + i(m)/m)^m - 1 = (1 + 0.072/2)^2 - 1 = 0.073296$$

The effective interest rate of 7%(12) is

$$i_2 = (1 + i(m)/m)^m - 1 = (1 + 0.07/12)^{12} - 1 = 0.072290$$

We conclude that 7.2%(2) is higher.

Problem 4, Page 87, exercise 16

How much additional interest is earned on a \$100,000 for three years if money is worth 6%(12) instead of 6%(1)?

Answer:

The principal is $P = \$100,000$, the term is three years.

First, we will calculate the amount if the money is worth 6%(12). In this case the nominal interest rate is 6%, there are 12 conversion periods per year, $m = 12$, and the total number of conversion periods in three years is $n = 12 \cdot 3 = 36$. Thus

$$S = \$100,000(1 + 0.06/12)^{36} = \$119,668.05$$

The interest earned is the difference between S and P. When the the money is worth 6%(22), the interest earned is $S - P = \$119,668.05 = \$19,668.05$.

Second, we will calculate the amount if the money is worth 6%(1). In this case the nominal interest rate is 6%, there is one conversion period per year, $m = 1$, and the total number of conversion periods in three years is $n = 1 \cdot 3 = 3$. Thus

$$S = \$100,000(1 + 0.06)^3 = \$119,101.6$$

The interest earned is the difference between S and P. When the the money is worth 6%(1), the interest earned is $S - P = \$119,101.6 = \$19,101.6$.

The additional interest earned on a \$100,000 for three years if money is worth 6%(2) instead of 6%(1) is $\$19,668.05 - \$19,101.6 = \$566.45$.

Problem 4, Page 87, exercise 20

Use the logic of exercise 19 to solve $8\%(4) = \_\_\%(2)$.

Answer:

If the money is compounded quarterly, then the length of the conversion period is 3 months. If the money is compounded semiannually, then then the length of the conversion period is 6 months. We can use a term of 6 months since both conversion lengths fit a whole number of times in the term.

First we need to calculate how much \$1 is worth in 6 months at $8\%(4)$.

$$S = P(1+i(m)/m)^n = \$1(1+0.08/4)^2 = \$1.0404$$

Next, given that \$1 is worth \$1.0404 in six months, we need to determine the nominal interest rate compounded semiannually. We need to solve the equation $\$1.0404 = \$1(1+i(2)/2)^1$. By subtracting 1 from both sides we have $0.0404 = i(2)/2$. By multiplying both sides by 2, we obtain $i(2) = 2 \cdot 0.0404 = 0.0808$. The equivalent nominal interest rate compounded semiannually is 8.08%.

Problem 4, Page 91, exercise 1

Compare the rate necessary to mature \$1 to \$3 in 5 years with the rate necessary to mature $\$1,000$ to $\$3,000$ in 5 years.

Answer:

To determine the effective rate necessary to mature \$1 to \$3 in 5 years we need to solve the equation

$$\$3 = \$1(1+i)^5$$

By multiplying both sides by $1,000$ we obtain

$$\$3,000 = \$1,000(1+i)^5$$

This is the equation we need to solve to determine the effective rate necessary to mature $\$1,000$ to $\$3,000$ in 5 years. We conclude the two rates are equal.

Note: This is true in general. Given the same term, when the principal and the amount are multiplied by the same number the effective interest rate stays the same.

Problem 4, Page 91, exercise 7

At what rate converted monthly is \$200 worth \$500 in 4 years? Express the answer as a rate per month and as a nominal rate.

Answer:

We are given $P = \$200$, $S = \$500$, $m = 12$, $t = 4$ years. Then the total number of conversion periods is $n = 4 \cdot 12 = 48$. We need to solve the equation

$$\$500 = \$200(1+i)^{48}$$

We divide both sides by \$200: $2.5 = (1+i)^4 8$. We raise both sides to the power $\frac{1}{48}$: $2.5^{\frac{1}{48}} = 1+i$. Finally, we subtract 1 from both sides: $i = 2.5^{\frac{1}{48}} - 1 = 0.0193 = 1.93\%$.

The interest rate per month of year is 1.93%. The nominal interest rate is $12 \cdot 1.93\% = 23.16\%$.

Problem 4, Page 91, exercise 10

At what rate converted quarterly could Michael double the value of his college fund in just 3 years? Express the answer as a rate per quarter and as a nominal rate.

Answer:

We don't know how much money Michael has now, let's denote them by P. Since Michael wants to double his money, then the amount is twice the principal, that is $S = 2P$. To determine the interest rate per half of year, i, we need to solve the equation

$$2P = P(1+i)^6$$

We divide by P and get $2 = (1+i)^1 2$. We raise both sides to the power $\frac{1}{12}$ and get $2^{\frac{1}{12}} = 1+i$. We conclude $i = 2^{\frac{1}{12}} - 1 = 0.05946 = 5.95\%$.

The interest rate per quarter is 5.95%. The nominal interest rate is $4 \cdot 5.95\% = 23.8\%$.

Problem 4, Page 95, exercise 10

If it takes 4.6 years for $\$2,000$ to mature to $\$3,500$, then, at the same rate, it will take 4.6 years for \$1 to mature to ______.

Answer:

The term is calculated as the total number of conversion periods times the length of a period. Since we assume the same rate for both cases, then we will have the same nominal interest rate and number of conversion periods per year, m. This implies we have the same interest rate per period, i, and the same total number of conversion periods n in both cases. We use the formula $n = \frac{\ln\left(\frac{S}{P}\right)}{\ln(1+i)}$:

$$\frac{\ln\left(\frac{\$3{,}500}{\$2{,}000}\right)}{\ln(1+i)} = \frac{\ln\left(\frac{S}{\$1}\right)}{\ln(1+i)}$$

We multiply both sides by $\ln(1+i)$: $\ln\left(\frac{\$3{,}500}{\$2{,}000}\right) = \ln(S)$ or $\ln(1.75) = \ln(S)$. Thus $S = \$1.75$.

Problem 4, Page 95, exercise 12

How long it will take $\$1,000$ to be worth $\$2,388$ at $8\%(4)$ if
a) We answer to the nearest (rounded) n?
b) No interest is given for a part of a period and worth at least $\$2,388$?
c) We use simple interest (Banker's Rule) for a part of a period?

Answer:

We have $P = \$1,000$, $S = \$2,388$, $m = 4$, $i(4) = 8\% = 0.08$. We use the formula

$$n = \frac{\ln\left(\frac{S}{P}\right)}{\ln(1+i)} = \frac{\ln\left(\frac{\$2,388}{\$1,000}\right)}{\ln(1+0.08/4)} = \frac{\ln(2.388)}{\ln(1.002)} = 43.96$$

Part a:

The nearest integer to 43.96 is 44.

Part b:

If no interest is given for a part of a period, then $n = 44$.

Part c:

After 43 conversion periods the $\$1,000$ is worth

$$\$1,000(1+0.08/4)^4 3 = \$2,343.18$$

To determine the part of a period needed for the $\$2,343.18$ to mature to $\$2,388$ we use simple interest.

$$\$2,388 \quad = \quad \$2,343.18\left(1+0.08\cdot\frac{x}{360}\right)$$

Then $x = 86$.

$\$1,000$ will be be worth $\$2,388$ at $8\%(4)$ in 43 quarters and 86 days.

Problem 4, Page 98, exercise 3

To analyze before investing, all pertinent information must be valued at the _____. Hence NPV and IRR have a focal date at _____.

Answer:

To analyze before investing, all pertinent information must be valued at the present. Hence NPV and IRR have a focal date at the present.

Problem 4, Page 98, exercise 8

Find Megan's payoff on 7/1/04 at 7% compounded semiannually if whe owes \$500 7/1/02 and on 1/1/03 she makes a \$200 payment and on 7/1/03 she makes another \$300 payment.

Answer:

The problem is asking us to determine how much has Megan to pay on 7/1/04 to settle the debt. To determine the amount Megan has left to pay we need to consider all the amounts at the date of the payoff, 7/1/04. I will consider the loan positive and the payments negative.

$$\$500(1+0.07/2)^4 - \$200(1+0.07/2)^3 - \$300(1+0.07/2)^2 = \$30.65$$

Problem 4, Page 98, exercise 14

A grandmother's trust fund of $\$250,000$ is to be distributed to her grandchildren, ages 5, 10, and 12. Her trust stipulates that each will get an equal amount in the year they each 18. Find that amount if money is earning $6\%(4)$.

Answer:

If x is the amount each grandchild receives, then the three payments considered at the present must add to $\$250,000$.

$$\$250,000 = x(1+0.06/4)^{-4\cdot 6} + x(1+0.06/4)^{-4\cdot 8} + x(1+0.06/4)^{-4\cdot 13}$$

Thus $x = \$140,323$.

Problem 4, Page 98, exercise 18

A $\$10,000$, 5-year note is sold in one year to a person requiring $7\%(4)$ on her money. Find the proceeds.

Answer:

The maturity value of the note is $\$10,000$ and the problem asks us to determine the sell price if the third party wants to make $7\%(4)$ on her money. The sell price is

$$P = S(1+i)^{-n} = \$10,000(1+0.07/4)^{-16} = \$7,576.16$$

Problem 4, Page 98, exercise 20

Find the NPV at $20\%(1)$ and $25\%(1)$ for an investment that requires a $\$5,000$ outlay of cash now to receive $\$4,000$ return in one year and $\$2,500$ return in two years.

Answer:

The net present value is the sum of investments (negative) and returns (positive) moved to the present.

$$NPV_{@20\%} = -\$5,000 + \$4,000(1+0.2)^{-1} + \$2,500(1+0.2)^{-2} = \$69.44$$

$$NPV_{@25\%} = -\$5,000 + \$4,000(1+0.25)^{-1} + \$2,500(1+0.25)^{-2} = -\$200$$

Problem Bonus 1

Solve the following exercises from textbook:
Page 92, exercise 21
Page 94, exercise 3

Answer:

Page 92, exercise 21

The compound amount formula when using discount interest for one conversion period is $P = S(1-d)^n$, where S is the amount, P is the principal, d is the discount rate per period and n is the total number of conversion periods. We divide both sides by S and then take the $\frac{1}{n}$ power of both sides:

$$\left(\frac{P}{S}\right)^{\frac{1}{n}} = 1-d$$

We derive the formula for the discount rate

$$d = 1 - \left(\frac{P}{S}\right)^{\frac{1}{n}}$$

We apply it to the data given in the problem and obtain $d = 1-(0.5)^{\frac{1}{7}} = 9.43\%$.

Page 94, exercise 3

First we need to determine the number of whole periods. We have the formula

$$n = \frac{\ln\left(\frac{S}{P}\right)}{\ln(1+i(m)/m)}$$

We denote by $\lfloor n \rfloor$ the floor of n, which is the largest integer smaller or equal to n. $\lfloor n \rfloor$ is the number of whole periods needed.

Next, we need to determine how much is the principal P worth after $\lfloor n \rfloor$ periods:

$$P(1+i(m)/m)^{\lfloor n \rfloor}$$

Now we need to determine the number of days needed for $P(1+i(m)/m)^{\lfloor n \rfloor}$ to worth S at simple interest $i(m)$. We denote the term by $t = \frac{x}{360}$ and we solve the following equation for x:

$$\begin{aligned}
S &= P(1+i(m)/m)^{\lfloor n \rfloor}\left(1+i(m)\cdot\frac{x}{360}\right) \\
1+i(m)\cdot\frac{x}{360} &= \frac{S}{P}(1+i(m)/m)^{-\lfloor n \rfloor} \\
i(m)\cdot\frac{x}{360} &= \frac{S}{P}(1+i(m)/m)^{-\lfloor n \rfloor} - 1 \\
x &= \frac{360}{i(m)}\left(\frac{S}{P}(1+i(m)/m)^{-\lfloor n \rfloor} - 1\right)
\end{aligned}$$

Problem (Bonus 2) Incomplete solution

Read the following article Probabalistic Auctions: Why Dont Universities Raffle off Chair Endowments?(the link is available on the class website).
a) Calculate the effective interest rate for Ben Franklin's investment. I am asking for the interest rate per conversion period for a compound interest loan with conversion period of one year.
b) Explain the 'concept' of 'probabilistic chair'.

Answer:

Ben Franklin donated $\$4,000$ to the Commonwealth of Pennsylvania in 1790. In 1990, his gift had grown to over $\$2,000,000$. Since we don't have an exact number we will assume it is exactly $\$2,000,000$. From 1790 to 1990 there are 200 years. To calculate the effective rate of interest we need to solve the equation

$$\$2,000,000 = \$4,000(1+i)^{200}$$

The solution is $i = 3.16\%$.

Problem (Bonus 3)

Read the following article Just the Facts: S&P's $2 Trillion Mistake and explain, in your own words, the mistake made by Standard and Poors (S&P) and the consequences of that mistake.

Answer:

Here is the part of the article that explains the mistake:

> Specifically, CBO calculated that the Budget Control Act, including its discretionary caps, would save $2.1 trillion relative to a baseline in which current discretionary funding levels grow with inflation.
>
> S&P incorrectly added that same $2.1 trillion in deficit reduction to an entirely different baseline where discretionary funding levels grow with nominal GDP over the next 10 years. Relative to this alternative baseline, the Budget Control Act will save more than $4 trillion over ten years or over $2 trillion more than S&P calculated. (The baseline in which discretionary spending grows with nominal GDP is substantially higher because CBO assumes that nominal GDP grows by just under 5 percent a year on average, while inflation is around 2.5 percent a year on average.

Problem (Bonus 4)

Alice takes a $\$180,000$ compound interest loan for 30 years, converted monthly,at an effective interest rate of 3%. Assuming a monthly payment of $758.89, calculate after how long more than half of the monthly payment goes to paying the principal. Find the minimal number of months.

Answer:

First we need to calculate the interest rate per month compounded monthly given the effective interest. We need to solve the equation

$$1+0.03 = (1+i)^12$$

We get $i = 0.0025$.

Solution 1

The principal right after the n^{th} payment, denoted by P_n is the original principal, P, minus all the payments, all quantities, R moved to the date of the n^{th} payment. Thus

$$\begin{aligned}
P_n &= P(1+i)^n - R(1+i)^{n-1} - R(1+i)^{n-2} - \cdots - R(1+i)^1 - R(1+i)^0 \\
&= P(1+i)^n - R\frac{(1+i)^n - 1}{1+i-1} \\
&= P(1+i)^n - \frac{R}{i}(1+i)^n + \frac{R}{i} \\
&= \left(P - \frac{R}{i}(1+i)^n\right) + \frac{R}{i}
\end{aligned}$$

The reduction in principal is $P_{n-1} - P_n$.

$$\begin{aligned}
P_{n-1} - P_n &= \left(P - \frac{R}{i}(1+i)^{n-1}\right) + \frac{R}{i} - \left(\left(P - \frac{R}{i}(1+i)^n\right) + \frac{R}{i}\right) \\
&= \left(P - \frac{R}{i}(1+i)^{n-1}\right)(1-(1+i)) \\
&= (R - iP)(1+i)^{n-1}
\end{aligned}$$

To determine after how long more than half of the monthly payment goes to paying the principal, we need to solve the inequality

$$(R-iP)(1+i)^{n-1} \geq \frac{R}{2}$$

Since we assume, that Alice is paying at least the interest, then $R - iP > 0$. Thus

$$(1+i)^{n-1} \geq \frac{R}{2(R-iP)}$$

Since ln is an increasing function,

$$(n-1)\ln(1+i) \geq \ln\left(\frac{R}{2(R-iP)}\right)$$

Since $1+i > 0$, then $\ln(1+i) > 0$, thus

$$n-1 \geq \frac{\ln\left(\frac{R}{2(R-iP)}\right)}{\ln(1+i)}$$

We conclude that

$$n \geq 1 + \frac{\ln\left(\frac{R}{2(R-iP)}\right)}{\ln(1+i)}$$

More than half of the monthly payment goes to paying the principal after 84 months.

Solution 2

Another solution is to use Excel. Type \$180,000 in the cell $A1$. In the cell $A2$ type $= A1*(1.0025) - 758.89$. In the cell $B2$ type $= A2 - A3$. Drag the cells $A2$ and $B2$ down. Thus, in the column A we have the current principal and in the column B we have the part of the payment that goes to the principal. Just look for the row for which the cell in the column B is bigger than $758.89/2 = 379.445$.

Solution 3 Use a computer program to solve the problem. Here is the Python code:

```
cp=180000
n=0
pp=cp
payment = 758.89
while (pp - cp < (payment / 2)):
   pp = cp
   cp = pp * (1 + 0.0025) - payment
   n = n + 1
print (n)
```