Financial Mathematics: Homework #7

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Problem 1

Dilbert makes annual deposits of \$5000 on the first day of each year for 20 years. If the effective rate of interest is 7%, how much is in the account immediately after the last deposit?

Answer:

We will use the maturity value formula of an ordinary annuity

$$S = R\frac{(1+i)^n - 1}{i}$$

where $R = \$5000$, $n = 20$, and $i = 7\%$.

$$S = \$5000\frac{(1+0.07)^{20} - 1}{0.07} = \$20,4977.46$$

Problem 2

On January 1, 2004 Ralph opens an IRA with a $\$2,000$ deposit. He continues to deposit $\$2,000$ at the beginning of each year until January 1, 2044, when he makes the final deposit. If the account earns an effective rate of interest of 9%, how much is in the account on the day of the last deposit?

Answer:

Since Ralph makes a deposit on January 1 from 2004 until 2040 (inclusive), the the number of payments is $n = 41$. The yearly payments is the rent, $R = \$2,000$. Since the time between two payments is one year, and the effective interest rate has also a period of one year, then $i = 0.09$. To determine the value on the day of the last deposit we use the maturity value formula for an ordinary annuity

$$S = R\frac{(1+i)^n - 1}{i}$$

to get

$$S = \$2000\frac{(1+0.09)^{41} - 1}{0.09} = \$738,583.73$$

Problem 3

Ralph deposits \$1000 on January 1 of each year from 1990 to 2015 into an account paying an effective rate of 6%. If there are no further deposits, how much is in the account on January 1, 2030?

Answer:

We will solve this problem in two steps: first, we calculate the value of the payments on the day of the last payment, second, we use the compound amount formula to move amount from the first step to 2030.

For step one we use the same reasoning as in the previous two problems. We have $R = \$1000$, $n = (2015 - 1990) + 1 = 26$, $i = 0.06$.
The value of the payments on the day of the last payment is

$$R\frac{(1+i)^n - 1}{i} = \$1000\frac{(1+0.06)^{26} - 1}{0.06} = \$59,156.38$$

Now, we use the compound interest formula $S = P(1+i)^n$ with $P = \$59,156.38$, $i = 0.06$, $n = 2030 - 2015 = 15$ (here n is the number of intervals, not the number of payments) to get the value of the account in January 1, 2030

$$S = \$59,156.38(1+0.06)^{15} = \$141,771.70$$

Problem 4

Luke makes deposits of \$50,000 on January 1 of years 2000, 2005, 2010, 2015, and 2020 into an account paying 10% interest convertible quarterly. How much is the account worth on January 1, 2040?

Answer:

This problem is easier to solve if we calculate the value of each payment on January 1, 2040 using the compound amount formula $S = P(1+i)^n$ since the interest rate is given as compound interest compounded quarterly and the rent period is five years.
The January 1, 2000 payment is worth $\$50,000(1+0.1/4)^{40\cdot 4}$ on January 1, 2040.
The January 1, 2005 payment is worth $\$50,000(1+0.1/4)^{35\cdot 4}$ on January 1, 2040.
The January 1, 2010 payment is worth $\$50,000(1+0.1/4)^{30\cdot 4}$ on January 1, 2040.
The January 1, 2015 payment is worth $\$50,000(1+0.1/4)^{25\cdot 4}$ on January 1, 2040.
The January 1, 2020 payment is worth $\$50,000(1+0.1/4)^{20\cdot 4}$ on January 1, 2040.
On January 1, 2040, the account is worth

$$\$50,000(1+0.1/4)^{40\cdot 4} + \$50,000(1+0.1/4)^{35\cdot 4} + \\ \$50,000(1+0.1/4)^{30\cdot 4} + \$50,000(1+0.1/4)^{25\cdot 4} + \\ \$50,000(1+0.1/4)^{20\cdot 4}$$

Problem 5

Suppose that an investment will make 20 annual payments of $\$12,000$, the first coming a year from now. Assuming an effective rate of 8%, what is the present value of the investment?

Answer:

The problem indicates that there are 20 annual payments, thus $n = 20$. The annual payment is $\$12,000$, thus $R = \$12,000$. The interest rate per year is the effective rate, thus $i = 0.08$.
The present value of an annuity is calculated at the beginning of the first rent period (at zero on the timeline we drew in class, the first payment is at one), and the first payment is a year from now. The present value of the investment is

$$P = R\frac{1-(1+i)^{-n}}{i} = \$12,000 \cdot \frac{1-(1+0.08)^{-20}}{0.08} = \$117,817.77$$

Problem 6

Suppose that an annuity pays $\$1,200$ once per year for 15 years, with the first payment coming one year from now. If the effective rate of interest is 9%, what is the present value?

Answer:

The period payment is $R = \$1,200$, the number of payments is $n = 15$, the interest rate per year is $i = 0.09$. The present value is

$$P = R\frac{1-(1+i)^{-n}}{i} = \$1,200 \cdot \frac{1-(1+0.09)^{-15}}{0.09} = \$9,672.83$$

Problem 7

An annuity will provide a sequence of 40 quarterly payments of $5000, the first coming 7 years from now. If we assume a nominal rate of 6% convertible monthly, what is the present value of the annuity?

Answer:

Since the interest rate is given for a loan converted monthly, but the rent period is one quarter (3 months), we need to calculate the equivalent interest rate with quarterly compounding. To do that we assume we have two accounts and we deposit $1, one with nominal interest rate $6\%(12)$ and one with unknown nominal interest rate $x(4)$. We keep the money in both accounts for one year. At the end of the year we have the same balance in both accounts. The balance of the account with nominal interest rate $6\%(12)$ is

$$(1+0.06/12)^1 2$$

The balance of the account with unknown nominal interest rate $x(4)$ is

$$(1+x/4)^4$$

Since these two balances are equal we have

$$(1+0.06/12)^12 = (1+x/4)^4$$

Now we need to solve this equation for x and get $x = 0.0603005$.

Since x is the nominal interest rate for a loan compounded quarterly, the interest rate per quarter is $i = x/4 = 0.015075125$

The first step is to use the present value of an ordinary annuity to calculate the value of the payments six years and three quarters from now:

$$R\frac{1-(1+i)^{-n}}{i} = \$5,000 \cdot \frac{1-(1+0.015075125)^{-40}}{0.015075125} = \$149,374.30589$$

Next, we use compound interest to move the amount above to the present. Th number of quarters in six years and three quarters is 27.

$$P = S(1+i)^{-n} = \$149,374.30589(1+0.015075125)^{-27} = \$99,729.78$$

Problem Bonus 1

Compare the Merchant's Rule and the US Rule for a compound interest loan, when one partial payment is made after a number of conversions periods.

Hint: Your solutions should have the following structure:
Step 1: Give names (letter labels) to all quantities in the problem: e.g the principal, the partial payment, the number of conversion periods until the partial payment, the number of conversion periods from the first payment to the due date.
Step 2: Calculate the balance using the Merchant's Rule.
Step 3: Calculate the balance using the US Rule.
Step 4: Compare the quantities at step 2 and 3. Might be easier is you subtract one from the other.

Answer:

Step 1:

Let P be the principal, Q be the payment, m the number of conversion periods until the first payment, and n the number of conversion periods from the first payment to the due date, i the interest rate per period.

Step 2:

The Merchant's Rule indicates the balance is the sum of the principal (negative) and the payments (positive) moved to the due date. The balance at the due date is

$$B_M = -P(1+i)^{n+m} + Q(1+i)^n$$

Step 3:

To calculate the balance using the US Rule, we must first calculate the balance at the first payment, which is the sum of the principal (negative) moved the first payment date and the first payment

$$-P(1+i)^m + Q$$

Since there are no more payments, all is left to do is to move the balance on the due date

$$B_{US} = (-P(1+i)^m + Q)(1+i)^n$$

Step 4:

$$\begin{aligned} B_{US} &= (-P(1+i)^m + Q)(1+i)^n \\ &= -P(1+i)^m(1+i)^n + Q(1+i)^n \qquad \text{(by distributivity)} \\ &= -P(1+i)^{n+m} + Q(1+i)^n \qquad \text{(by one of the power laws)} \\ &= B_M \end{aligned}$$

Problem (Bonus 2)

Consider an annuity for which the payments are made at the beginning of the rent period. Suppose the annuity has n payments. Calculate the value of the annuity at the end of the last rent period.

Hint: Given the number of payments n, the rent or payment R, the interest rate per rent period i. Your answer should be a formula containing these symbols.

Answer:

First payment accrues interest for $n+1$ rent periods, thus its value at the end of the last period is $R(1+i)^{n+1}$.
Second payment accrues interest for n rent periods, thus its value at the end of the last period is $R(1+i)^n$.
Continue. The last payment accrues interest for one conversion period, thus its value at the end of the last period is $R(1+i)$.

The value of the annuity at the end of the last rent period is the sum of all the payments moved to the end of the last period

$$\begin{aligned} R(1+i)^{n+1} &+ R(1+i)^n + \cdots + R(1+i) \\ &= R(1+i)((1+i)^n + (1+i)^{n-1} + \cdots + (1+i) + 1) \\ &= R(1+i)\frac{(1+i)^n - 1}{(1+i) - 1} \\ &= R(1+i)\frac{(1+i)^n - 1}{i} \end{aligned}$$

Problem (Bonus 3)

Consider an annuity for which the payments are made at the beginning of the rent period. Suppose the annuity has n payments. Calculate the value of the annuity at the beginning of the first rent period.

Hint: Given the number of payments n, the rent or payment R, the interest rate per rent period i. Your answer should be a formula containing these symbols.

Answer:

The first payment is already at the beginning of the first rent period.
The second payment moved to the beginning of the first rent period is $R(1+i)^{-1}$.
The third payment moved to the beginning of the first rent period is $R(1+i)^{-2}$.
Continue.
The last payment moved to the beginning of the first rent period is $R(1+i)^{-(n-1)}$.
The value of all the payments at the beginning of the first rent period is the sum of all the payments moved at the beginning of the first rent period:

$$\begin{aligned} R &+ R(1+i)^{-1} + R(1+i)^{-2} + \cdots + R(1+i)^{-(n-1)} \\ &= R[1 + (1+i)^{-1} + (1+i)^{-2} + \cdots + (1+i)^{-(n-1)}] \\ &= R\left(1 + \frac{1}{1+i} + \left(\frac{1}{1+i}\right)^2 + \cdots + \left(\frac{1}{1+i}\right)^{(n-1)}\right) \\ &= R\frac{\left(\frac{1}{1+i}\right)^n - 1}{\frac{1}{1+i} - 1} \\ &= R\frac{(1+i)^{-n} - 1}{(1+i)^{-1} - 1} \end{aligned}$$

Problem (Bonus 4)

Assume Alice has saved $\$5,000$ and wants to open a retirement account. She has two options:
(1) pay $x\%$ tax now, and not pay tax when she withdraws money
(2) not pay tax now, deposit the entire $\$5,000$, but pay $y\%$ tax on the money she withdraws.
Assuming that both accounts will earn interest with an effective rate of 6%, discuss which option is more advantageous for Alice. Your answer should depend on wheter x or y is bigger.

Answer:

(1) If Alice pays $x\%$ tax now, then Alice has $(1-x)\$5,000$ left to invest. In n years, Alice will have $((1-x)\$5,000)(1+0.06)^n = (1-x)\$5,000(1+0.06)^n$.

(2) If Alice invests the entire amount today, in n years the $\$5,000$ will be worth $\$5,000(1+i)^n$. After Alice pays $y\%$ tax, she has left $(1-y)(\$5,000(1+i)^n) = (1-y)\$5,000(1+i)^n$.

If $x = y$ then the two options are identical.

If $x < y$, then $-y < -x$. It follows $1-y < 1-x$. By multiplying both sides with $\$5,000(1+i)^n$, we get $(1-y)\$5,000(1+i)^n < (1-x)\$5,000(1+i)^n$. We conclude that option 2 is less advantageous than option 1.

Similarly, if $y < x$, then option 2 is more advantageous.