Homework 8 - Solutions

Math 1140 Financial Mathematics

Collaboration Policy: You are encouraged to collaborate with your fellow students on this homework. You must turn in individual solutions and you are not allowed to use any written, typed, or recorded artifact from the meeting with your classmates. You are allowed to use any resources **except for the Appendix D in the textbook (the solutions to the odd-numbered exercises)**.

Pledge: On my honor, I pledge that I have neither given nor received unauthorized aid on this assignment.

Name(use block letters):

Signature:

For full credit you must show your work and your calculations for all the problems. I am not asking for the presentation of silly arithmetic!

Problem1. Use the geometric series formula

$$1+x+x^2+\cdots+x^n=\frac{x^{n+1}-1}{x-1}$$

to calculate or give the formula for:
a) $1+x+x^2+\cdots+x^{n+1}$

Answer: The geometric sum formula works for all positive natural numbers n, in particular it works for $n+1$.

$$1+x+x^2+\cdots+x^{n+1}=\frac{x^{(n+1)+1}-1}{x-1}=\frac{x^{n+2}-1}{x-1}$$

b) $1+x+x^2+\cdots+x^{n-1}$

Answer: The geometric sum formula works for all positive natural numbers n, in particular it works for $n-1$, when $n\geq 2$.

$$1+x+x^2+\cdots+x^{n-1}=\frac{x^{(n-1)+1}-1}{x-1}=\frac{x^{n}-1}{x-1}$$

c) $x+x^2+\cdots+x^n$

Answer: We observe that the term 1 of the sum is missing. We will use distributivity and the power law $x^n\cdot x^m=x^{n+m}$

$$x+x^2+\cdots+x^n=x(1+x+\cdots+x^{n-1})$$

The sum in the parenthesis is exactly the sum from part b. Using the formula above we have

$$x + x^2 + \cdots + x^n = x \cdot \frac{x^n - 1}{x - 1}$$

d) $1 + x^3 + x^6 + x^9 + \cdots + x^{3n}$

Answer: We can not apply the formula directly, because the powers are not increasing by one. We observe that the powers of consecutive terms increase by 3. We use another power rule, $(x^n)^m = x^{n \cdot m}$ to rewrite the sum as

$$\begin{aligned} 1 + x^3 + x^6 + x^9 + \cdots + x^{3n} &= 1 + x^{1\cdot 3} + x^{2\cdot 3} + x^{3\cdot 3} + \cdots + x^{3\cdot n} \\ &= 1 + (x^3)^1 + (x^3)^2 + (x^3)^3 + \cdots + (x^3)^n \end{aligned}$$

We can apply the geometric sum formula with x replaced by x^3

$$\begin{aligned} 1 + x^3 + x^6 + x^9 + \cdots + x^{3n} &= \frac{(x^3)^{n+1} - 1}{x^3 - 1} \\ &= \frac{x^{3(n+1)} - 1}{x^3 - 1} \end{aligned}$$

e) $x^{-1} + x^{-2} + \cdots + x^{-n}$

Answer: First we transform the negative powers in fractions using $x^{-k} = x^{(-1)\cdot k} = (x^{-1})^k = \left(\frac{1}{x}\right)^k$

$$x^{-1} + x^{-2} + \cdots + x^{-n} = \left(\frac{1}{x}\right)^1 + \left(\frac{1}{x}\right)^2 + \cdots + \left(\frac{1}{x}\right)^n$$

Now we can apply the formula from part c replacing x with $\frac{1}{x}$

$$\begin{aligned} x^{-1} + x^{-2} + \cdots + x^{-n} &= \frac{1}{x} \cdot \frac{\left(\frac{1}{x}\right)^n - 1}{\frac{1}{x} - 1} \\ &= \frac{x^{-n} - 1}{1 - x} \end{aligned}$$

f) $x^{-2} + x^{-4} + \cdots + x^{-2n}$

Answer: We will try use the formula from part e.

$$x^{-2} + x^{-4} + \cdots + x^{-2n} = (x^2)^{-1} + (x^2)^{-2} + \cdots + (x^2)^{-n}$$

We replace x with x^2 in the formula in part e and we obtain the answer.

Problem2. Alice borrows P dollars at an interest rate i per month. Assume Alice makes a monthly payment, R. Is there a problem if the monthly payment R is less than the interest per month, iP?

Answer: If the monthly payments do not cover the interest, the debt continues to grow as the unpaid part of the interest adds to the balance each month. If Alice is not aware of this and she thinks she is slowly paying off the debt, then this is not a good situation for her. If Alice uses this money for an investment opportunity that will bring her returns high enough to pay off the debt then there is no problem.

Problem3. Assume a $\$20,000$ credit card debt and an effective rate of 15%.
a) What is the minimum yearly payment that will cover the interest?

Answer: The minimum yearly payment that will cover the interest is equal to the interest

$$\$20,000 \cdot 0.15$$

b) If the yearly payment does not exceed this minimum, what happens to the debt?

Answer: The debt will continue to grow.

Problem4. Jan borrows $\$40,000$ at $12\%(2)$ for her college education and wants to repay it within ten years by making semiannual payments. Find the semiannual payment.

Answer: The $\$40,000$ Jan borrows represent the present value of the payments she will make over ten years. To determine the payment we will use the present value formula

$$P = R\frac{1-(1+i)^{-n}}{i}$$

Thus

$$R = \frac{iP}{1-(1+i)^{-n}}$$

The interest rate per rent period is $i = 0.12/2 = 0.06$. Since the term is ten years and there are two payments per year, then $n = 20$.

$$R = \frac{0.06 \cdot \$40,000}{1-(1.06)^{-20}} = \$3,487.38$$

Problem5. Emil is buying new appliances for his kitchen and finds a double oven which costs $800. After a $200 down payment it can financed over one year with monthly payments at a rate of only $1.4\%(12)$. Find Emil's monthly payments.

Answer: Since Emil makes a $200 down payment, the he borrows only $600. The monthly payment is calculated using the same formula as in the previous problem. In this case, the interest rate per rent period is $i = 0.014/12$, and $n = 12$.

$$R = \frac{0.014/12 \cdot \$600}{1-(1+0.014/12)^{-12}} = \$50.38$$

Problem6. How many quarters will it take a deposit if \$100 per quarter at $6\%(4)$ to :
a) to accumulate close to $\$5,000$, but not go over (payments less than \$100 are not allowed)

Answer: The $\$5,000$ represent the maturity value of the \$100 payments per quarter. We will use the formula:

$$S = R\frac{(1+i)^n - 1}{i}$$

We solve for n and we have

$$n = \frac{\ln\left(\frac{iS}{R} + 1\right)}{\ln(1+i)}$$

Since $n = 37.6$, then it will take 37 quarters to accumulate close to $\$5,000$, but not go over.

b) to accumulate at least $\$5,000$ (payments less than \$100 are not allowed)

Answer: It will take 38 quarters to accumulate at least $\$5,000$ (payments less than \$100 are not allowed).

c) to accumulate exactly $\$5,000$, the last payment will be less than \$100. Find the final payment.

Answer: If 37 deposits of \$100 are made, the account balance will be slightly smaller than $\$5,000$. If 38 deposits are made then the account balance is over $\$5,000$. The solution to have a balance of exactly $\$5,000$ is to make 37 payments of \$100 and a final payment smaller than \$100. The value of the first 37 deposits of \$100 at the 38th quarter is

$$S = R\frac{(1+i)^n - 1}{i}(1+i) = \$100\frac{(1+0.06/4)^{37} - 1}{0.06/4}(1+0.06/4) = \$4971.99$$

The last payment is $\$5,000 - \$4971.99 = \$28.01$.

Problem7. How long will it take Alice if she is depositing \$250 per month into an account paying $5.1\%(12)$ to accumulate $\$15,000$ for buying a new car?

Answer: We use the formula from the previous problem, with $R = \$250$, $i = 0.051/12$, $S = \$15,000$.

Problem8. At what rate of interest will \$100 per month be worth $\$2,500$ in two years?

Answer: The interest rate per month is 0.353%. The nominal interest rate compounded monthly is $12 \cdot 0.353\% = 4.236\%$.

Problem9. I you buy a $\$10,000$ car by making 24 \$500 monthly payments, what is the rate you are being charged?

Answer: The interest rate per month is 1.212%. The nominal interest rate compunded monthly is $14.544\%(12)$.

Problem10. What is the APR for an 8% add-on loan for $\$3,000$ for three years?

Answer: To determine the monthly payment of an add-on loan we calculate the maturity value using the simple interest formula $S = P(1+it)$ and then divide by the number of months in the term. Thus

$$R = \frac{\$3,000(1+0.08\cdot 3)}{3\cdot 12} = \$103.33$$

To determine the interest rate per month, we plug in the values in Wolfram Alpha as described in class: 1.288%. The APR is $12\cdot 1.288\% = 15.456\%$.

Problem11. A $\$9,000$ car is purchased using a 4% add-on loan and monthly payments over three years. What is the actual APR? To determine the monthly payment of an add-on loan we calculate the maturity value using the simple interest formula $S = P(1+it)$ and then divide by the number of months in the term. Thus

$$R = \frac{\$9,000(1+0.04\cdot 3)}{3\cdot 12} = \$280$$

To determine the interest rate per month, we plug in the values in Wolfram Alpha as described in class: 0.6259%. The APR is $12\cdot 0.6259 = 7.5108\%$.

ProblemBonus 1. A $\$10,000$ loan for 36 months at $6\%(12)$ is arranged for the costumer to make payments R for the first year, $2R$ for the second year, and $3R$ for the third year. Find the payments for each of the three years.

Answer: We only learned formulas for annuities with equal payments, but we can split the payments above in:
1. one ordinary annuity with 12 payments equal to R.
2. one deferred annuity with 12 payments equal to $2R$, with the first payment after 13 months
3. one deferred annuity with 12 payments equal to $3R$, with the first payment after 25 months

We use the present value formula of an ordinary annuity for the first annuity

$$P = R\frac{1-(1+i)^{-n}}{i} = R\frac{1-(1+0.06/12)^{-12}}{0.06/12} = 11.6189R$$

The present value of the next twelve payments is calculated as the present value of an ordinary annuity moved back 12 conversion periods.

$$P = 2R\frac{1-(1+i)^{-n}}{i}(1+i)^{-m} = R\frac{1-(1+0.06/12)^{-12}}{0.06/12}(1+0.06/12)^{-12} = 21.8879R$$

The present value of the last twelve payments is calculated as the present value of an ordinary annuity moved back 24 conversion periods.

$$P = 3R\frac{1-(1+i)^{-n}}{i}(1+i)^{-m} = R\frac{1-(1+0.06/12)^{-12}}{0.06/12}(1+0.06/12)^{-24} = 30.9245R$$

The present value of all the payments is their sum $11.6189R + 21.8879R + 30.9245R = 64.4313R$. The present value of the payments it is also the amount borrowed, thus $\$10,000 = 64.4313R$, $R = \$155.2041$.
The first year payments are $R = \$155.2041$. The second year payments are $2R = 2 * \$155.2041 = \$310.4082$. The third year payments are $3R = 3 * \$155.2041 = \$465.6123$.

Problem(Bonus 2). John wants to save $\$28,000$ in eight years. He plans to deposit $\$2,000$ per year for five years and $\$3,000$ per year for the last three years. What rate will he need to be able to have the desired amount?

Answer: We use the unknown interest rate i to calculate the maturity value of the payments. The maturity value of the first five payments at the time of the last payment is given by the formula

$$S = R\frac{(1+i)^n - 1}{i} = \$2,000\frac{(1+i)^5 - 1}{i}$$

To determine the value of the first five payments 3 years later we use the compound amount formula $S = P(1+i)^n$ and we get

$$\$2,000\frac{(1+i)^5 - 1}{i}(1+i)^3$$

The value of the last three payments at the last payment is

$$S = R\frac{(1+i)^n - 1}{i} = \$3,000\frac{(1+i)^3 - 1}{i}$$

The sum of all the payments moved to the last payment is $\$28,000$. The equation we need to solve is

$$\$2,000\frac{(1+i)^5 - 1}{i}(1+i)^3 + \$3,000\frac{(1+i)^3 - 1}{i} = \$28,000$$

To solve this equation type

solve 28000=2000((1+x)^5-1)/x(1+x)^3 + 3000((1+x)3^3-1)/x*

in Wolfram Alpha and you will get the answer $x = 0.1203$.