Homework 9 - Solutions

Math 1140 Financial Mathematics

Collaboration Policy: You are encouraged to collaborate with your fellow students on this homework. You must turn in individual solutions and you are not allowed to use any written, typed, or recorded artifact from the meeting with your classmates. You are allowed to use any resources **except for the Appendix D in the textbook (the solutions to the odd-numbered exercises)**.

Pledge: On my honor, I pledge that I have neither given nor received unauthorized aid on this assignment.

Name(use block letters):

Signature:

For full credit you must show your work and your calculations for all the problems. I am not asking for the presentation of silly arithmetic!

Problem1. Find the present value of an ordinary annuity that lasts five years and pays $\$3,000$ at the end of each month, using a nominal interest rate of 3% convertible monthly. Then repeat the problem using an annual effective discount rate of 3%. Which is higher? Why?

Answer: First, I will solve the problem using the nominal interest rate of 3% convertible monthly. Since the annuity lasts five years and the rent period is one month, then the number of rent periods is $n = 5 \cdot 12 = 60$. Since the annuity earns a nominal interest rate of 3% convertible monthly, then the interest rate per month is $i = 0.03/12 = 0.0025$. Since the annuity pays $\$3,000$ at the end of each month, the the rent is $R = \$3,000$. All wee need to do now is to apply the present value formula for an ordinary annuity

$$P = R\frac{1-(1+i)^{-n}}{i} = \$3,000\frac{1-(1+0.0025)^{-60}}{0.0025} = \$166,957.07$$

Next, I will solve the problem using the effective discount rate of 3%. The compounding period for the effective interest rate is one year. Since the annuity pays the rent monthly, then we need to calculate the equivalent interest rate compounded monthly. To determine the equivalent interest rate we calculate the maturity value of \$1 for one year, first using the known discount rate $3\%(1)$, and second using the unknown interest rate per month, x. Then we set the two maturity values equal to each other and we solve for x.

The maturity value of \$1 for one year at a discount rate of $3\%(1)$ is $\$1(1-0.03)^{-1}$. The number of compounding periods is one, because the term is one year and the conversion period is also one year.

The maturity value of \$1 for one year at interest rate x per month compounded monthly is $\$1(1+x)^{12}$. The number of compounding periods is one, because the term is one year and the conversion period is one month.

Now we solve for x the equation $\$1(1-0.03)^{-1} = \$1(1+x)^{12}$. First we raise both sides to the power $\frac{1}{12}$

$$\$1(1-0.03)^{-\frac{1}{12}} = \$1(1+x)$$

Now, we subtract 1 from both sides and we get $x = (1-0.03)^{\frac{1}{12}} - 1 = 0.00254$.

The only difference from the previous case is the interest rate.

$$P = R\frac{1-(1+i)^{-n}}{i} = \$3,000\frac{1-(1+0.00254)^{-60}}{0.00254} = \$166,754.09$$

The present value decreases when the interest rate per conversion period increases. The reason is that when money earns more interest you need fewer money to start with. The interest rate per rent period is lower in the first case, 0.250%, than in the second case, 0.254%, therefore the present value is bigger in the first case.

Problem2. Mrs Williams finds that she has two options for investing $\$32,000.02$ for fifteen years.
Option 1: Deposit the $\$32,000.02$ into a fund earning a nominal rate of discount $d(4)$ convertible quarterly.
Option 2: Buy an ordinary annuity with 15 equal payments, the annuity payments computed using an annual effective rate 7%, and then, when she gets an annuity payment, she invests it immediately into a fund earning an annual effective rate of 5%.
Mrs Williams calculates that the second option produces an accumulated value that is $\$1,500$ more than the accumulated value yielded by the first option.
Calculate $d(4)$.

Answer: The maturity value of option 1 is

$$S_1 = P(1-d(m)/m)^{-n}$$

Since $\$32,000.02$ is deposited now, then this is the present value $P = \$32,000.02$. Since nominal rate of discount $d(4)$ convertible quarterly, then $m = 4$.

Now we need to determine the number of conversion periods. Since in the option 2 we have 15 annual payments, then the term is 15 years. Thus, the number of conversion periods is $n = 4 \cdot 15 = 60$.

Therefore

$$S_1 = \$32,000.02(1-d(4)/4)^{-60}$$

Option 2 is a little bit more complicated. First we need to calculate the rent payment of the ordinary annuity with 15 equal payments, the annuity payments computed using an annual effective rate 7%. The present value of the annuity is $P = \$32,000.02$. The number of payments is $n = 15$. The interest rate per rent period is $i = 7\%$.

$$R = \frac{iP}{1-(1+i)^{-n}} = \frac{0.07 \cdot \$32,000.02}{1-(1+0.07)^{-15}} = \$3,514.42$$

Mrs Williams invests each annuity payment immediately into a fund earning an annual effective rate of 5%. This is a sequence of equal payments, in other words an annuity. To calculate the value of the investment we use the maturity value formula of an ordinary annuity

$$S_2 = R\frac{(1+i)^n - 1}{i} = \$3,514.42\frac{(1+0.05)^{15} - 1}{0.05} = \$75,814.78$$

The second option produces an accumulated value that is $\$1,500$ more than the accumulated value yielded by the first option means $S_1 + \$1,500 = S_2$. To determine $d(4)$ we need to solve the equation

$$\$32,000.02(1 - d(4)/4)^{60} + \$1,500 = \$75,814.78$$

First we subtract $\$1,500$ from both sides

$$\$32,000.02(1 - d(4)/4)^{60} = \$74,314.78$$

Next, we divide both sides by $\$32,000.02$

$$(1 - d(4)/4)^{-60} = 2.32$$

Next, we raise both sides to the power $-\frac{1}{60}$

$$1 - d(4)/4 = 2.32^{-\frac{1}{60}} = 0.9861$$

Next, we subtract 1 from both sides

$$-d(4)/4 = -1 + 0.9861 = -0.0139$$

Next, we multiply both sides by -4

$$d(4) = 0.0556$$

We conclude that the discount compound rate is $5.56\%(4)$ convertible quarterly.

Problem3. April received an inheritance from her grandmother in the form of an annuity. The annuity pays $\$3,000$ on Jan 1 from 1966 trough 1984. Find the value of this annuity on Jan 1, 1966 using an annual effective interest rate of 5%.

Answer: First, we need to determine the type of annuity. Since the first payment is on Jan 1, 1966 and we are asked to calculate the value of the rent payments on Jan 1, 1966, then we should use the formula for an annuity due:

$$P = R\frac{1 - (1+i)^{-n}}{i}(1+i)$$

The rent payment is $R = \$3,000$. The interest rate per rent period is $i = 0.05$. The number of payments is $1984 - 1966 + 1 = 19$. We need to add one because the term of the annuity-due runs from the first payment, Jan 1, 1966, trough one year after the last payment, Jan 1, 1985.

$$P = \$3,000\frac{1 - (1+0.05)^{-19}}{0.05}(1+0.05) = \$38,068.76$$

Problem4. Suppose the interest rate is 3% per month. Find the value one month before the first payment of an annuity-due paying \$200 at the beginnig of each month for five years.

Answer: This annuity makes $n = 12 \cdot 5 = 60$ payments. The interest rate per conversion period is $i = 0.03$.

We are asked to calculate the value of the annuity one conversion period before the first payment. There are two ways of solving this problem.

First solution

The present value of an annuity-due is calculated at the time of the first payment and we use the formula

$$P = R\frac{1-(1+i)^{-n}}{i}(1+i)$$

The problem is asking us to calculate the value one conversion period before the first payment. We use the present value formula for compound interest $P = S(1+i)^{-n}$ where S is the present value of the annuity-due, $n = 1$. Thus, the value of the annuity-due one conversion period before the first payment is

$$R\frac{1-(1+i)^{-n}}{i}(1+i)(1+i)^{-1} = R\frac{1-(1+i)^{-n}}{i}$$

Second solution

The present value one conversion period before the first payment of an annuity-due is the present value of an ordinary annuity with the same rent and number of payments. Thus, the value of the annuity-due one conversion period before the first payment is

$$R\frac{1-(1+i)^{-n}}{i} = \$200\frac{1-(1+0.03)^{-60}}{0.03} = \$5,535.11$$

Problem5. Steve Wong wishes to save for his retirement by depositing $\$1,200$ at the beginning of each year for thirty years. Exactly one year after his last deposit he wishes to begin making annual equal withdrawals until he has made twenty withdrawals and exhausted the savings. Find the amount of each withdrawal if the effective rate is 5% during the first thirty years but only 4% after that.

Answer: First we must understand where exactly on the timeline is the interest change. The interest rate of 5% is earned for the first thirty years. The deposits are made at the beginning of each year for thirty years. The interest rate of 5% is earned for a term starting with the first payment and ends one year after the last payment. This is an annuity-due. The value of all the deposits after thirty years is

$$S = R\frac{(1+i)^n-1}{i}(1+i) = \$1,200\frac{(1+0.05)^{30}-1}{0.05}(1+0.05) = \$83,712.94$$

The withdrawals start one year after the last payment. The beginning of the term of this annuity starts at the same time as the previous one ends. The interest rate for the period of 20 years corresponding to the twenty withdrawals is 4%. The maturity value of the deposits becomes the present value of an annuity-due consisting of the twenty withdrawls. To determine the rent, R, we use the present value formula of an annuity-due

$$P = R\frac{1-(1+i)^{-n}}{i}(1+i)$$

$$P = R\frac{1-(1+i)^{-n}}{i}(1+i)$$

$$R = \frac{iP}{1-(1+i)^{-n}}(1+i)^{-1} = \frac{0.04 \cdot \$83,712.94}{1-(1+0.04)^{-20}}(1+0.04)^{-1} = \$5,922.83$$

Problem6. Alice owned an annuity which had equal payments for twelve consecutive years, the first of these being in exactly twelve years. She sold it, and the selling price of $\$21,092.04$ was based on an yield rate for the investor of $7.8\%(1)$. What is the amount of the equal payments?

Answer: The sell price of a financial instrument is calculated based on the maturity value of the financial instrument and the interest rate required by the third party. We solved this type of problem for promissory notes, but the idea is the same. A financial instrument, in this case deferred annuity, is sold to a third party requiring a certain interest rate. In most of the problems we were asked to calculate the sell price. This problem is a little bit more difficult. We can pretend we know the value of R and we will calculate the sell price as a function of R. Since the problem is giving us the sell price, then set the previous function equal to $\$21,092.04$ and we solve for R

The first payment of the annuity is in exactly twelve years and there are twelve payments, thus the last payment is in exactly twenty-three years. The term of the annuity is 23 years. This is also the term the third-party has his investment.

The maturity value of the annuity does not depend on the twelve years before the payments start and we can use the formula for ordinary annuity since the last payment is at the end of the term

$$S = R\frac{(1+i)^n-1}{i} = R\frac{(1+0.078)^{12}-1}{0.078}$$

The sell price if the third-party requires a rate of $7.8\%(1)$ is calculated using the present value formula for compound interest $P = S(1+i)^{-n}$

$$P = R\frac{(1+0.078)^{12}-1}{0.078}(1+0.078)^{-23}$$

The selling price is $\$21,092.04$. Therefore, to determine R we need to solve the equation

$$\$21,092.04 = R\frac{(1+0.078)^{12}-1}{0.078}(1+0.078)^{-23}$$

We obtain $R = \$21,092.04 \cdot (1+0.078)^{23} \cdot \frac{0.078}{(1+0.078)^{12}-1} = \$6,328.00$