Homework 10 - Solutions

Math 1140 Financial Mathematics

Collaboration Policy: You are encouraged to collaborate with your fellow students on this homework. You must turn in individual solutions and you are not allowed to use any written, typed, or recorded artifact from the meeting with your classmates. You are allowed to use any resources **except for the Appendix D in the textbook (the solutions to the odd-numbered exercises)**.

Pledge: On my honor, I pledge that I have neither given nor received unauthorized aid on this assignment.

Name(use block letters):

Signature:

For full credit you must show your work and your calculations for all the problems. I am not asking for the presentation of silly arithmetic!

Problem1. To secure a 20-year $9\%(12)$ loan on their $\$200,000$ home, a couple must pay 10% down and 3.2 points which will be added to the loan.
a) Find the amount of the loan.

Answer: Since the down payment is 10%, then the couple needs to borrow the rest of 90% of $\$200,000$, that is $\$180,000$.
Since the points are added to the loan, the amount borrowed is

$$\frac{\$180,000}{1-0.032} = \$185,950.41$$

b) Find the payment.

Answer: The monthly payment is

$$R = \frac{iP}{1-(1+i)^{-n}} = \frac{\frac{0.09}{12}\cdot \$185,950.41}{1-(1+\frac{0.09}{12})^{-12\cdot 20}} = \$1673.05$$

c) Find the actual APR.

Answer: To calculate the actual APR we use Wolfram Alpha or any financial calculator with present value equal to $\$180,000$, periodic payment equal to $\$1673.05$ and number of payments equal to $12\cdot 20$, and we get an interest rate of $9.46\%(12)$.

Problem2. To secure a 15-year 10.3%(12) loan on his \$300,000 home, Albert must pay 20% down and 2 points.
a) Find the amount of the loan.

Answer: Since the points are not added to the loan and the down payments is 20%, then the amount of the loan is \$240,000 .

b) Find the payment.

Answer: The monthly payment is

$$R = \frac{iP}{1-(1+i)^{-n}} = \frac{\frac{0.103}{12} \cdot \$240,000}{1-(1+\frac{0.103}{12})^{-12\cdot 15}} = \$2623.27$$

c) Find the actual APR.

Answer: To calculate the actual APR we use Wolfram Alpha or any financial calculator with present value equal to $\$240,000(1-0.02) = \$235,200$, periodic payment equal to \$2623.27 and number of payments equal to $12 \cdot 15$, and we get an interest rate of 10.66%(12).

Problem3. Assuming points are financed over the term of the loan, would you do better with a 20-year 6%(12) \$100,000 loan or a 20-year 5.8%(12) \$100,000 loan with 2 points? Why is no loan value needed to make the comparison?

Answer: The monthly payments of the 20-year 6%(12) \$100,000 loan is

$$R = \frac{iP}{1-(1+i)^{-n}} = \frac{\frac{0.06}{12} \cdot \$100,000}{1-(1+\frac{0.06}{12})^{-12\cdot 20}} = \$716.44$$

The monthly payments of the 20-year 5.8%(12) \$100,000 loan with 2 points is

$$R = \frac{i\frac{P}{1-p}}{1-(1+i)^{-n}} = \frac{\frac{0.058}{12} \cdot \frac{\$100,000}{1-0.02}}{1-(1+\frac{0.058}{12})^{-12\cdot 20}} = \$719.33$$

The first loan gives a smaller monthly payment, thus being the better loan.
Both monthly payments can be expresses as a multiple of the amount financed. Which one is smaller depends only on the constant we multiply the amount financed by.

Problem4. Assuming points are financed over the term of the loan, would you do better with a 25-year 8%(12) \$200,000 loan with 2 points or a 25-year 7.9%(12) \$200,000 loan with 4 points?

Answer: The monthly payment of the 25-year 8%(12) \$200,000 loan with 2 points is

$$R = \frac{i\frac{P}{1-p}}{1-(1+i)^{-n}} = \frac{\frac{0.08}{12}\cdot\frac{\$200{,}000}{1-0.02}}{1-(1+\frac{0.08}{12})^{-12\cdot 25}} = \$1,575.14$$

The monthly payment of the 25-year 7.9%(12) \$200,000 loan with 4 points is

$$R = \frac{i\frac{P}{1-p}}{1-(1+i)^{-n}} = \frac{\frac{0.079}{12}\cdot\frac{\$200{,}000}{1-0.04}}{1-(1+\frac{0.079}{12})^{-12\cdot 25}} = \$1,594.18$$

The first loan gives a smaller monthly payment, thus being the better loan.

Problem5. Find the outstanding balance at the end of 6 years for a \$55,600 loan financed at 7.5%(12) for 15 years.

Answer: First, we need to calculate the monthly payment

$$R = \frac{iP}{1-(1+i)^{-n}} = \frac{\frac{0.075}{12}\cdot\$55,600}{1-(1+\frac{0.075}{12})^{-12\cdot 15}} = \$515.42$$

The outstanding balance is calculated using *prospective method* as the principal minus the payments already made, all quantities considered at the end of 6 years.

$$P(1+i)^k - R\frac{(1+i)^k-1}{i} = \$55,600\left(1+\frac{0.075}{12}\right)^{6\cdot 12} - \$515.42\cdot\frac{(1+\frac{0.075}{12})^{6\cdot 12}-1}{\frac{0.075}{12}}$$

The outstanding balance at the end of 6 years is \$40,390.01.

Problem6. A home costing \$115,800 was financed over 30 years with 20% down at 8.8%(12) and 1.25 points. If the points are added to the loan, what is the outstanding balance after 16 years and 8 months?

Answer: The amount needed is $(1-0.2)\$115,800 = \$92,640$. Since the points are added to the loan, the amount borrowed is $\frac{\$92{,}640}{1-0.0125} = \$93,812.66$.
Next, we need to calculate the monthly payment

$$R = \frac{iP}{1-(1+i)^{-n}} = \frac{\frac{0.088}{12}\cdot\$93,812.66}{1-(1+\frac{0.088}{12})^{-12\cdot 30}} = \$741.38$$

The outstanding balance is calculated using *retrospective method*

$$P(1+i)^k - R\frac{(1+i)^k-1}{i} = \$93,812.66\left(1+\frac{0.088}{12}\right)^{16\cdot 12+8} - \$741.38\cdot\frac{(1+\frac{0.088}{12})^{16\cdot 12+8}-1}{\frac{0.088}{12}}$$

The outstanding balance at the end of 16 years and 8 months is \$69,688.75.

Problem7. To secure a 20-year $9\%(12)$ loan on their $\$200,000$ home, a couple must pay 10% down and 3.2 points that will be added to the loan.
a) Find the outstanding balance after 5 years.

Answer: The amount needed is $(1-0.1)\$200,000 = \$180,000$. Since the points are added to the loan, the amount borrowed is $\frac{\$180,00}{1-0.032} = \$185,950.41$.
Next, we need to calculate the monthly payment

$$R = \frac{iP}{1-(1+i)^{-n}} = \frac{\frac{0.09}{12}\cdot\$185,950.41}{1-(1+\frac{0.09}{12})^{-12\cdot 20}} = \$1,673.05$$

The outstanding balance is calculated using *retrospective method*

$$P(1+i)^k - R\frac{(1+i)^k-1}{i} = \$185,950.41\left(1+\frac{0.09}{12}\right)^{5\cdot 12} - \$1,673.05\cdot\frac{(1+\frac{0.09}{12})^{5\cdot 12}-1}{\frac{0.09}{12}}$$

The outstanding balance at the end of 5 years is $\$164,950.68$.

b) If they can refinance after 5 years at $7\%(12)$ with no points, keeping the same total term, what would their new payment be?

Answer: There are 25 years left in the term. Thus, the term of the new loan is 15 years. The amount borrowed of the new loan is the outstanding balance of the old loan The monthly payment of the new loan is

$$R = \frac{iP}{1-(1+i)^{-n}} = \frac{\frac{0.07}{12}\cdot\$164,950.68}{1-(1+\frac{0.07}{12})^{-12\cdot 15}} = \$1,482.63$$

c) Find the total savings in b because refinancing.

Answer: The monthly savings are $\$1,673.05 - \$1,482.63 = \$190.42$. The total savings are $12\cdot 15\cdot \$190.42 = \$34275.6$.

d) If they can refinance after 5 years at $7\%(12)$ with no points, changing the new term to 10 years, what would their new payment be?

Answer: The monthly payment of the new loan is

$$R = \frac{iP}{1-(1+i)^{-n}} = \frac{\frac{0.07}{12}\cdot\$164,950.68}{1-(1+\frac{0.07}{12})^{-12\cdot 10}} = \$1,915.21$$

e) Find the total savings in d because refinancing.

Answer: The total savings because refinancing are $12\cdot 15\cdot \$1,673.05 - 12\cdot 10\cdot \$1,915.21 = \$71,323.8$.

ProblemBonus 1. If a 15-year loan for $\$40,000$ is financed at $8\%(12)$ with 1.75 points, how many points would a person have to pay to create an equivalent APR on a loan that is financed at $7.5\%(12)$?

Answer: First we calculate the APR of the $8\%(12)$ loan. The monthly payment of the $8\%(12)$ loan is

$$R = \frac{i\frac{P}{1-p}}{1-(1+i)^{-n}} = \frac{\frac{0.08}{12} \cdot \frac{\$40,000}{1-0.0175}}{1-(1+\frac{0.08}{12})^{-12 \cdot 15}} = \$389.07$$

To calculate the actual APR we use Wolfram Alpha or any financial calculator with present value equal to $\$40,000$, periodic payment equal to $\$389.07$ and number of payments equal to $12 \cdot 15$, and we get an interest rate of $9.46\%(12)$.
The monthly payment of the $7.5\%(12)$ loan is

$$R = \frac{i\frac{P}{1-p}}{1-(1+i)^{-n}} = \frac{\frac{0.075}{12} \cdot \frac{\$40,000}{1-x}}{1-(1+\frac{0.075}{12})^{-12 \cdot 15}}$$

Since both loans have the same actual APR and the same present value, then the monthly payments are the same. We solve for x the equation

$$\$389.07 = \frac{i\frac{P}{1-p}}{1-(1+i)^{-n}} = \frac{\frac{0.075}{12} \cdot \frac{\$40,000}{1-x}}{1-(1+\frac{0.075}{12})^{-12 \cdot 15}}$$

The solution is 4.695 points.

ProblemBonus 2. Compare the outstanding balance calculated using the prospective and retrospective method. Consider a loan of P dollars for n months with interest rate i per month. Compare the outstanding balance after the k^{th} payment using the prospective method with the outstanding balance after the k^{th} payment using the retrospective method.

Answer: The outstanding balance after the k^{th} payment calculated using the prospective method is given by the formula

$$OB_R = R\frac{1-(1+i)^{-(n-k)}}{i}$$

The outstanding balance after the k^{th} payment calculated using the retrospective method is given by the formula

$$\begin{aligned}
OB_P &= P(1+i)^k - R\frac{(1+i)^k-1}{i} \\
&= R\frac{1-(1+i)^{-n}}{i}(1+i)^k - R\frac{(1+i)^k-1}{i} \\
&= R\left(\frac{1-(1+i)^{-n}}{i}(1+i)^k - \frac{(1+i)^k-1}{i}\right) \\
&= R\left(\frac{(1+i)^k-(1+i)^{-n+k}}{i} - \frac{(1+i)^k-1}{i}\right) \\
&= R\frac{(1+i)^k-(1+i)^{-n+k}-(1+i)^k+1}{i} \\
&= R\frac{1-(1+i)^{-(n-k)}}{i} \\
&= OB_R
\end{aligned}$$

ProblemBonus 3. The Jackson family has a loan for $\$82,000$ financed at $8\%(12)$ for 30 years. At the end of the seventh year they start paying an additional \$70 per month toward the principal of the loan. If they continue to do this what will the new term be and what will their last payment be?

Answer: The monthly payment is

$$R = \frac{iP}{1-(1+i)^{-n}} = \frac{\frac{0.08}{12}\cdot \$82,000}{1-(1+\frac{0.08}{12})^{-12\cdot 30}} = \$601.69$$

The outstanding balance at the end of the seventh year is

$$\begin{aligned}
P(1+i)^k - R\frac{(1+i)^k-1}{i} &= \$82,000\left(1+\frac{0.08}{12}\right)^{7\cdot 12} - \$601.69\cdot\frac{(1+\frac{0.08}{12})^{7\cdot 12}-1}{\frac{0.08}{12}} \\
&= \$75,831.15
\end{aligned}$$

The number of payments is determined by solving for n the equation

$$\$75,831.15 = (\$601.69+\$70)\frac{1-(1+\frac{0.08}{12})^{-n}}{\frac{0.08}{12}}$$

The solution is $n = 210.2$. The number of full payments is 210.
To determine the last partial payment, we solve for x the equation

$$\$75,831.15 = (\$601.69+\$70)\frac{1-(1+\frac{0.08}{12})^{-210}}{\frac{0.08}{12}} + x\left(1+\frac{0.08}{12}\right)^{-211}$$

ProblemBonus 4. Use outstanding balance to determine how much of the k^{th} payment of a mortgage goes to the interest.

Answer: The outstanding balance is the principal left to be paid. The interest part of the k^{th} monthly payment is i times the principal left to be paid after the previous payment:

$$i\left(P(1+i)^{k-1} - R\frac{(1+i)^{k-1}-1}{i}\right) = iP(1+i)^{k-1} - R((1+i)^{k-1}-1)$$

ProblemBonus 5. A schoolteacher arranges for his $\$50,000$ home mortgage to be paid with 9 payments each year from October 1 trough June 1, but no payments during the 3 months of the summer. Find the payment if this is a 15-year mortgage at $9\%(12)$.

Answer: We don't have a formula for calculating the periodic payment of an annuity with unequal payments. First, we consider the nine payments as a forborne annuity with three conversion periods after the last payment. The maturity value of such an annuity is

$$S = R\frac{(1+i)^n-1}{i}(1+i)^p = R\frac{(1+\frac{0.09}{12})^9-1}{\frac{0.09}{12}}\left(1+\frac{0.09}{12}\right)^3$$

If we replace each of the fifteen forborne annuities with the corresponding maturity value, we have an ordinary annuity with fifteen annual payments. We need to calculate the effective interest rate equivalent to $9\%(12)$.

$$i_e = \left(1+\frac{0.09}{12}\right)^{12} - 1$$

The present value of this new ordinary annuity is $\$50,000$.

$$\$50,000 = \left(R\frac{(1+\frac{0.09}{12})^9-1}{\frac{0.09}{12}}\left(1+\frac{0.09}{12}\right)^3\right)\frac{1-((1+\frac{0.09}{12})^{12})^{-15}}{(1+\frac{0.09}{12})^{12}-1}$$