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Signature: ____

There are 75 minutes for this exam and 100 points on the test; don't spend too long on any one question!

The 12 short answer questions require only a sentence or two for full credit; the three long answer questions have their own page, and obviously require more. The questions are organized by topic, so the long answer questions are scattered throughout the exam (questions 5, 14, and 15). The long answer questions are worth about half of the test score; the short answer questions are all worth 4 points each, and constitute the other half. The reference sheet is on page 2.

All work must be on these exam pages.

Good luck!

Part I: Logic	_____ / 33
Part II: Structures	_____ / 16
Part III: Proofs	_____ / 51
Total	_____ / 100

CS 202

Exam 1 Reference Sheet

Set and logical identities

Sets (Rosen, p. 89)	Name	Boolean logic (Rosen, p. 24)
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws	$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws	$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$
$A \cup A = A$ $A \cap A = A$	Idempotent laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
$\overline{(\overline{A})} = A$	Complementation law	$\neg(\neg p) \equiv p$
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	DeMorgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws	$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$

Rules of inference (Rosen, p. 58)

Rule of Inference	Tautology	Name
$\underline{p}$ $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
p $\underline{q}$ $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
p $\underline{p \rightarrow q}$ $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$\neg q$ $\underline{p \rightarrow q}$ $\therefore \neg p$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \vee q$ $\underline{\neg p}$ $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Disjunctive syllogism
$p \vee q$ $\underline{\neg p \vee r}$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

Part I: Logic

Question 1 (4 points): What is the converse of $p \rightarrow q$?

Answer: The converse is $q \rightarrow p$

Question 2 (4 points): Given the Boolean proposition $p \leftrightarrow q$, write an equivalent compound proposition using only the operators $\neg$, $\wedge$, and $\vee$.

Answer: $(\neg p \vee q) \wedge (\neg q \vee p)$

Question 3 (4 points): State the negation of the following quantified statement: $\forall x \exists y (P(x) \wedge \neg Q(y))$

Answer: $\exists x \forall y (\neg P(x) \vee Q(y))$

Question 4 (4 points): What are the two ways to convert a propositional function into a proposition?

Answer: Either by supplying it with a constant, or by adding a quantifier (for each variable)

Question 5 (17 points): Prove that $(p \wedge (p \wedge q)) \wedge (\neg p \vee q) \equiv (p \wedge q)$ using logical equivalences. You must clearly label each step of the logical equivalence.

Answer:

$(p \wedge (p \wedge q)) \wedge (\neg p \vee q) \equiv (p \wedge q)$	Original statement
$(p \wedge p \wedge q) \wedge (\neg p \vee q) \equiv (p \wedge q)$	Associativity of AND (removing of parenthesis)
$(p \wedge q) \wedge (\neg p \vee q) \equiv (p \wedge q)$	Idempotent law
$((p \wedge q) \wedge \neg p) \vee ((p \wedge q) \wedge q) \equiv (p \wedge q)$	Distributive law
$(q \wedge p \wedge \neg p) \vee (p \wedge q \wedge q) \equiv (p \wedge q)$	Associativity of AND (removing of parenthesis)
$(q \wedge F) \vee (p \wedge q \wedge q) \equiv (p \wedge q)$	Negation law
$F \vee (p \wedge q \wedge q) \equiv (p \wedge q)$	Domination law
$(p \wedge q \wedge q) \equiv (p \wedge q)$	Identity law
$(p \wedge q) \equiv (p \wedge q)$	Idempotent law

Grimaldi, question 7(a), section 2.2, page 66.

Part II: Structures (sets and functions)

Question 6 (4 points): What is the difference between a subset and a proper subset?

Answer: A subset can be equal to the original set; a proper subset cannot be (it must have fewer elements than the original set)

Question 7 (4 points): Why must a function *f* be 1-to-1 and onto if the function *f* is invertible (i.e. you can find an inverse function of *f*)?

Answer: The function must be 1-to-1, as otherwise the inverse function has multiple answers for a single input. The function must be onto, as otherwise the inverse function is not defined for certain values.

Question 8 (4 points): What is the cardinality of a power set of a set of *n* elements?

Answer: 2^n

Question 9 (4 points): Let $f(x) = 5x + 2$ and $g(x) = 2x + 3$. What is $(f \circ g)(x)$?

Answer: $(f \circ g)(x) = 5(2x+3) + 2 = 10x + 17$

Part III: Proofs

Question 10 (4 points): What two properties must be shown for a uniqueness proof?

Answer: Existence and uniqueness

Question 11 (4 points): Write an existential generalization of the quantified statement $\exists x P(x)$. If you introduce new variables, etc., clearly describe what they represent.

Answer: $P(y)$, where y is some (unknown) constant, not a variable

Question 12 (4 points): What is the difference between a vacuous proof and a trivial proof?

Answer: A vacuous proof is when the antecedent of the conditional is always false, and thus the conditional is always true. A trivial proof is when the consequence of the conditional is always true, and thus the conditional is always true.

Question 13 (4 points): What movie did Professor Bloomfield show a preview of during class?

Answer: A parody trailer of Star Wars: Episode III

Question 14 (15 points): For this question, you will have to prove that if m is an even integer, then m+7 is an odd integer. You need to prove it two different ways: by direct proof, indirect proof, or proof by contradiction. Each proof method is worth the same amount.

Answer:

Proof method 1 (circle one): **Direct proof** Indirect proof Proof by contradiction

The statement translates into $p \rightarrow q$, where p means m is even, and q means m+7 is odd. For the direct proof, we assume p is true, and show that q must always be true. If m is even, then $m=2k$, where k is some integer (this is the definition of even numbers). Thus, $m+7 = 2k+7 = 2(k+3)+1$, where k+3 is an integer (as k was an integer). This is the definition of odd numbers (two times an integer plus one). Thus, m+7 must be odd.

Proof method 2 (circle one): Direct proof **Indirect proof** Proof by contradiction

The statement translates into $p \rightarrow q$, where p means m is even, and q means m+7 is odd. For the indirect proof, we prove the contrapositive. The contrapositive is $\neg q \rightarrow \neg p$, which translate to "if m+7 is even, then m is odd". If m+7 is even, then $m+7 = 2k+1$, where k is some integer (definition of odd numbers). Solving for m, we get $m = 2k-6 = 2(k-3)$, where k-3 is an integer (as k was an integer). This is the definition of even numbers (two times another integer). Thus, m must be even.

Proof method 2 (circle one): Direct proof Indirect proof **Proof by contradiction**

The statement translates into $p \rightarrow q$, where p means m is even, and q means m+7 is odd. For the proof by contradiction, we assume that the statement is false, and arrive at a contradiction. For the statement to be false, we assume that p must be true, and q must be false. Thus, we assume that m is even, and m+7 is even. If m is even, then $m=2k$, where k is some integer (this is the definition of even numbers). Thus, $m+7 = 2k+7 = 2(k+3)+1$. This, however, is the definition of odd numbers, and therefore m+7 must be odd. This contradicts our assumption that m+7 is even. Thus, the statement must be true, as the one case where it is false (i.e. when m is even, and m+7 is even) cannot occur.

Grimaldi, section 2.5, theorem 2.4, pages 114-115

Question 15 (20 points): Consider the following statements.

1. If Dominic goes to the racetrack, then Helen will be mad.
2. If Ralph plays cards all night, then Carmela will be mad.
3. If either Helen or Carmela gets mad, then Veronica (their attorney) will be notified.
4. Veronica has not heard from either of these two clients.

From these, can we conclude the following?

- Dominic didn't make it to the racetrack and Ralph didn't play cards all night.

Write each of these statements in symbolic form. Clearly label what your Boolean variables represent! Then establish the validity of the conclusion. You must clearly label which rule of inference is used for each step.

Answer:

r = Dominic goes to the racetrack
h = Helen gets mad
p = Ralph plays cards all night
c = Carmela gets mad
v = Veronica is notified

1. $r \rightarrow h$
2. $p \rightarrow c$
3. $h \vee c \rightarrow v$
4. $\neg v$

- $\neg r \wedge \neg p$

1.	$h \vee c \rightarrow v$	3rd hypothesis
2.	$\neg v$	4th hypothesis
3.	$\neg(h \vee c)$	Modus tollens on steps 1 and 2
4.	$\neg h \wedge \neg c$	DeMorgan's law on step 3
5.	$\neg h$	Simplification of step 4
6.	$r \rightarrow h$	1st hypothesis
7.	$\neg r$	Modus tollens on steps 5 and 6
8.	$\neg c$	Simplification of step 4
9.	$p \rightarrow c$	2nd hypothesis
10.	$\neg p$	Modus tollens on steps 8 and 9
11.	$\neg r \wedge \neg p$	Conjunction on steps 7 and 10

Grimaldi, question 12(b), section 2.3, page 86.