Mar 22 Slides

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Agenda

DeMorgan’s ?’s

Quiz Connection

Integers

Fundamental Theorem of Arithmetic

Proof by Contradiction Template

Example 1: Irrationals

Example 2:

DeMorgan's Questions

$$\neg(P \lor Q) \equiv \underline{\hspace{4cm}}$$

$$\neg(A \land B \land C) \equiv \underline{\hspace{4cm}}$$

DeMorgan’s Questions

$$\neg(P \lor Q) \equiv \neg P \land \neg Q$$

$$\neg(A \land B \land C) \equiv \neg A \lor \neg B \lor \neg C$$

DeMorgan's Questions

$$\neg \forall x P(x) \equiv$$

$$\neg \exists x P(x) \equiv$$

DeMorgan's Questions

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

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No x in the domain can make $f(x) = -6$

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$$\nexists\, x \in \mathbb{Z}.\ (\ f(x) = -6\ )$$

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why is this not enough??

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However, this is not the case for f(x), because there are members of its co-domain that are not part of the function's range; for example, -6 is in the co-domain but not the range.

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Suppose f(x) *is* a surjective function.

By definition, a function is surjective if its co-domain is the same set as its range.

However, this is not the case for f(x), because there are members of its co-domain that are not part of the function's range; for example, -6 is in the co-domain but not the range.

Therein lies the contradiction. Therefore, f(x) is not a surjective function. ∎

Quiz Question

General structure of proof by contradictions

We'll start by using some examples from **number theory (the study of integers).**

Intro to Integers

What’s a factor?

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That is, for some integer ***k***

$$ak = b$$

Let’s list the factors of 4!

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1, 2, 4, -4, -2, -1

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From this point forward, we will only be referring to the positive integers though.

What’s a prime

A prime is a number greater than 1 that is divisible only by 1 and itself.

Is 2 a prime number?

“Fundamental theorem of Arithmetic”

Aka Unique Factorization thm

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60 = 2*2*3*5

^The one and only way of expressing 60 as a set of prime factors

“Prime factorization” -> multiplicity of factor 2 is 2 (of 5 is 1)

“Fundamental theorem of Arithmetic”

Aka Unique Factorization thm

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Greatest Common Divisor

A **common divisor** of a and b is a number that divides them both.

The **greatest common divisor** of a and b is the biggest number that divides them both.

gcd(60, 8) = ???

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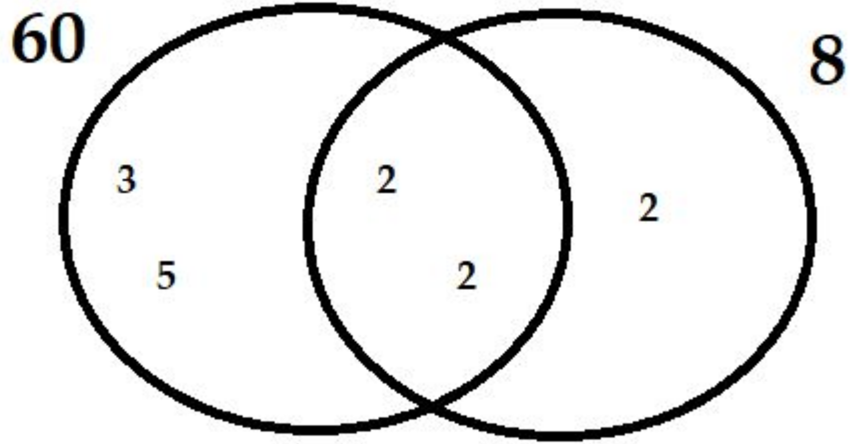
“intersect”



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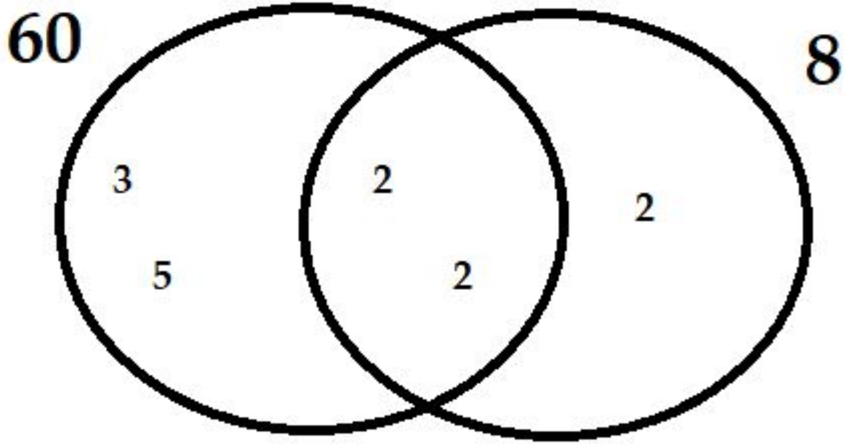
What is gcd(90, 84) = ?



Relatively Prime

Two positive integers greater than 1 are **relatively prime** iff gcd(x, y) = 1

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Collatz conjecture: Start with any positive integer n. Then, if the number is even, divide by 2. If it's odd, multiply by 3 and add 1. Repeat. The conjecture is that no matter what value of n, the sequence will always reach 1.

Proof by Contradiction

Proof by contradiction

Theorem: Not A

We proceed by contradiction.

Assume A to prove this”

...

A proves FALSE

Therein lies the contradiction.

Therefore

Not A

Often how we prove something is impossible in computing

Proof by contradiction

Thm: ½ is not an element of the natural numbers

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Proof by contradiction -- Informal

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One side has zero factors, one has at least 1?? Contradiction

Proof by contradiction -- Formal

Thm: $\frac{1}{2}$ is not an element of the natural numbers

We proceed by contradiction
Assume $\frac{1}{2} \in \mathbb{N}$.
Then, $\exists x \in \mathbb{N}.\ x = \frac{1}{2}$
By algebra, that means $2x = 1$.

By the fundamental theorem of arithmetic, both sides of the equation are equal, so 1 and 2x must have the same unique prime factorization.

But the factors of 2x include 2, and the factors of 1 do not. Therein lies the contradiction.

Therefore, [Because this assumption led to a contradiction,] $\frac{1}{2} \notin \mathbb{N}$

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Rational Numbers -- $\mathbb{Q}$

Represented by $\mathbb{Q}$, which stands for quotient

That's because all rational numbers can be represented by a quotient of two integers! aka an improper (or proper) fraction…

$x \in \mathbb{Q}$ iff $x = \frac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ and are relatively prime.

Proof by Contradiction -- $\mathbb{Q}$

$x \in \mathbb{Q}$ iff $x = \dfrac{a}{b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$ and are relatively prime.

Prove by contradiction that √2 is not a rational number.