Mar 31 Slides

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Agenda

Quiz Notes

Proof by Contradiction-- Longer Example 1

Proof by Contradiction-- Longer Example 2

Paper and Autograded Quiz

- Proof by contradiction: only 30 minutes to write
- Prime and relatively prime numbers
- Prime factorization and Fundamental Theorem of Algebra
- GCD

Proof by contradiction

Theorem: A is true.

We proceed by contradiction.

Assume **Not A** to prove this

...

Not A proves FALSE

Therein lies the contradiction.

Therefore

A is true.

Often how we prove something is impossible in computing

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Proof by contradiction

Prove $\sqrt[2]{2}\sqrt[3]{3}$ *is not a rational number*

We proceed by contradiction:

Suppose that $\sqrt[2]{2}\sqrt[3]{3}$ *is* a rational number.

Proof by contradiction

$$Prove\ \sqrt[2]{2}\sqrt[3]{3}\ is\ not\ a\ rational\ number$$

We proceed by contradiction:

Suppose that $\sqrt[2]{2}\sqrt[3]{3}$ *is* a rational number.

By the definition of rational numbers:

$$x \in \mathbb{Q} \text{ iff } x = \frac{a}{b} \quad \text{where } a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}$$

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So $\sqrt[2]{2}\sqrt[3]{3}$ can be expressed as a fraction of 2 integers.

Proof by contradiction

What next?

Proof by contradiction

What next?

First idea: Square both sides, to get:

$$2\sqrt[3]{3} = \frac{a^2}{b^2}$$

Proof by contradiction

Second idea: Raise both sides to the 6th power.

$$(\sqrt[2]{2}\sqrt[3]{3})^6 \quad = (\tfrac{a}{b})^6 \ = \ 2^3 3^2 \ = \ 72$$

Proof by contradiction

Second idea: Raise both sides to the 6th power.

$$(\sqrt[2]{2}\sqrt[3]{3})^6 = (\tfrac{a}{b})^6 = 2^3 3^2 = 72$$

By algebra, we get:

$$a^6 = 72b^6$$

Proof by contradiction

Third Idea: Keep in prime factor form.

$$a^6 = 2^3 3^2 b^6$$

Proof by contradiction

Third Idea: Keep in prime factor form.

$a^6 = 2^3 3^2 b^6$

Now, we know that $a^6 = 2^3 3^2 b$ must have an odd multiplicity of 2's in a^6's prime factorization.

Proof by contradiction

But this is impossible because a number raised to an even power (6) must have even multiplicities of all its factors.

Proof by contradiction

But this is impossible because a number raised to an even power (6) must have even multiplicities of all its factors.

Therefore, assuming $\sqrt[2]{2}\sqrt[3]{3}$ is a rational number led to a contradiction, we have proven that $\sqrt[2]{2}\sqrt[3]{3}$ is not a rational number. ■

Proof by contradiction

Proof by contradiction

$$Prove \ \frac{1+\sqrt{5}}{2} \ is\ not\ a\ rational\ number$$

Proof by contradiction

Prove $\frac{1+\sqrt{5}}{2}$ *is not a rational number*

We proceed by contradiction:

Proof by contradiction

$Prove\ \frac{1+\sqrt{5}}{2}\ is\ not\ a\ rational\ number$

We proceed by contradiction:

Suppose that $\frac{1+\sqrt{5}}{2}$ *is* a rational number.

Proof by contradiction

Prove $\frac{1+\sqrt{5}}{2}$ *is not a rational number*

We proceed by contradiction:

Suppose that $\frac{1+\sqrt{5}}{2}$ *is* a rational number.

By the definition of rational numbers:

Proof by contradiction

Prove $\frac{1+\sqrt{5}}{2}$ *is not a rational number*

We proceed by contradiction:

Suppose that $\frac{1+\sqrt{5}}{2}$ *is* a rational number.

By the definition of rational numbers:

$$x \in \mathbb{Q} \text{ iff } x = \frac{a}{b} \quad \text{where } a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}$$

Proof by contradiction

Prove $\frac{1+\sqrt{5}}{2}$ *is not a rational number*

We proceed by contradiction:

Suppose that $\frac{1+\sqrt{5}}{2}$ *is* a rational number.

By the definition of rational numbers:

$$x \in \mathbb{Q} \text{ iff } x = \frac{a}{b} \text{ where } a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}$$

So $\frac{1+\sqrt{5}}{2}$ can be expressed as a fraction of 2 integers.

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So $\frac{1+\sqrt{5}}{2}$ can be expressed as a fraction of 2 integers.

What next?

Proof by contradiction

By arithmetic, $\frac{1+\sqrt{5}}{2}$ can be broken up into two fractions:

$$\frac{1}{2} + \frac{\sqrt{5}}{2}$$

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By observation, the first fraction, $\frac{1}{2}$, is a rational number, since $1 = a$ and $2 = b$.

Proof by contradiction

Since we know that adding an irrational number to a rational number will result in an irrational number, we can now focus on determining if the second fraction, $\frac{\sqrt{5}}{2}$, is rational.

Proof by contradiction

$\frac{\sqrt{5}}{2}$ can be expressed as a fraction of 2 integers if it is rational:

$$\frac{\sqrt{5}}{2} = \frac{a}{b}$$

Proof by contradiction

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Squaring both sides, we get:

$$\frac{5}{4} = \frac{a^2}{b^2}$$

Proof by contradiction

$\frac{\sqrt{5}}{2}$ can be expressed as a fraction of 2 integers if it is rational:

$$\frac{\sqrt{5}}{2} = \frac{a}{b}$$

Squaring both sides, we get:

$$\frac{5}{4} = \frac{a^2}{b^2}$$

By algebra, $5b^2 = 2^2a^2$

Proof by contradiction

Now, we know that $5b^2 = 2^2a^2$ must have the same prime factorization by the fundamental theorem of algebra. So, we must have an odd multiplicity of 5's in 2^2a^2's prime factorization.

Proof by contradiction

Now, we know that $5b^2 = 2^2a^2$ must have the same prime factorization by the fundamental theorem of algebra. So, we must have an odd multiplicity of 5's in 2^2a^2's prime factorization.

But this is impossible because a number raised to an even power (2) must have even multiplicities of all its factors.

Proof by contradiction

Therefore, assuming $\frac{1+\sqrt{5}}{2}$ is a rational number led to a contradiction, and we have proven that $\frac{1+\sqrt{5}}{2}$ is not a rational number. ■

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