Oct 21st Slides

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Proof by contradiction

Theorem: Not A

We proceed by contradiction.

Assume A is true

...

A proves FALSE

Therein lies the contradiction.

Therefore

Not A

Often how we prove something is impossible in computing

Rational Numbers -- $\mathbb{Q}$

Represented by $\mathbb{Q}$, which stands for quotient

That's because all rational numbers can be represented by a quotient of two integers! aka an improper (or proper) fraction…

$$x \in \mathbb{Q} \text{ iff } x = \frac{a}{b} \quad \text{where } a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}^{+}$$

Proof by Contradiction -- $\mathbb{Q}$

Prove by contradiction that $\sqrt{2}$ is not a rational number.

Proof by Contradiction -- $\mathbb{Q}$

Prove by contradiction that $\sqrt{2}$ is not a rational number.

We will proceed by contradiction.

Assume that $\sqrt{2}$ is a rational number. Then, $\sqrt{2}$ can be expressed as a fraction n/d where $n \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$.

Squaring both sides gives $2=n^2/d^2$ and so $2d^2=n^2$
By the fundamental theorem of arithmetic, the prime factorization of $2d^2$ and n^2 must be equal. The term $2d^2$ must include the factor 2 with an *odd* multiplicity. However, all the prime factors of n^2 must have an *even* multiplicity (ie, all the factors of *n* twice). Therein lies the contradiction.

Therefore, $\sqrt{2}$ is not a rational number. ▮

Well-Ordering Principle

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Seems obvious but useful in proofs yet to come!

Prove that all the natural numbers are *finite*

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Assume that some natural numbers are infinite.

Let $F \subseteq \mathbb{N}$ be the all of the infinite natural numbers. By our assumption, |F| > 0.

By the well-ordering principle, there exists a smallest infinite number, $x \in F$. Then, let's create a new variable that is x subtract 1, $z = x$-2 = y-1. Since y is smaller than x, this means that y is finite.

However, subtracting 1 from an infinite number results in another infinite number, meaning x-1 = y cannot be finite. Therein lies the contradiction.

Therefore, all the natural numbers are finite.

Prove that the cardinality of natural numbers is infinite.

We will proceed by contradiction.

Assume that the cardinality of the natural numbers is finite (not infinite).

Consider the greatest element of $\mathbb{N}$, called x.

Let $y = x+1$.

Because adding two natural numbers results in a natural number, $y \in \mathbb{N}$ that is greater than x. However, x was the largest natural number. Therein lies the contradiction.

Therefore, there are an infinite number of natural numbers.

Prove that there are an infinite number of natural numbers.

An infinite-sized set of finite values: important in Theory of Computation.

More practice:

Website practice

Textbook Chapter 2.