September 30 Slides

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∃ and ∀

Associate "for all" with AND's since it becomes false if just one truth value is false

Associate "there exists" with OR's since it becomes true if just one truth value is true

Last Class:

Domain: People   L(x, y) = *x loves y*

Are these equivalent?

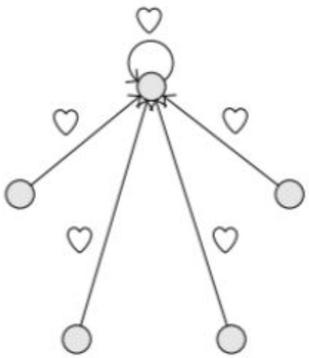
$$\exists y \forall x\, L(x,y) \overset{?}{\equiv} \forall x \exists y\, L(x,y)$$

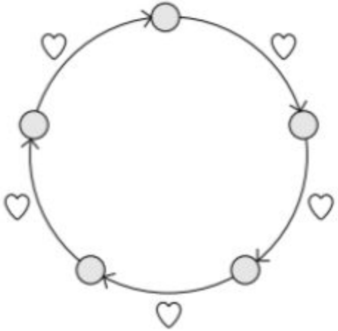
Quick Intro to Multiple Quantifiers:

Domain: People   L(x, y) = *x loves y*

Are these equivalent?

$\exists y \forall x\, L(x,y)$ **is not equivalent to** $\forall x \exists y\, L(x,y)$





Quick Intro to Multiple Quantifiers:

Domain: People   L(x, y) = *x loves y*

Are these equivalent?

$\exists y \forall x\, L(x,y)$ **is not equivalent to** $\forall x \exists y\, L(x,y)$

Remember

When you are dealing with mixed quantifiers, the order is very important. $\forall x \exists y\, R(x,y)$ is not logically equivalent to $\exists y \forall x\, R(x,y)$.

Think about nested loops

Domain: {Ann, Bob, Chris} $\exists y \forall x\, L(x,y)$

```
// since ∃ means stuff “or’d” together, start with false
existValue = False
for y in {Ann, Bob, Chris}:
    // since ∀ means stuff “and’d” together, start with true
    univValue = True
    for x in {Ann, Bob, Chris}:
        univValue = univValue ∧ L(x,y)
    end
    existValue = existValue ∨ univValue
end
Return existValue
```

Think about nested loops

Domain: {Ann, Bob, Chris} $\exists y \forall x\, L(x,y)$

How will this code change for "$\forall x \exists y\, L(x,y)$"?

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existValue = False
for y in {Ann, Bob, Chris}:
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```

Think about nested loops

Domain: {Ann, Bob, Chris} $\forall x \exists y\, L(x,y)$

```
// since ∀ means stuff "and'd" together, start with true
univValue = True
for x in {Ann, Bob, Chris}:
    // since ∃ means stuff "or'd" together, start with false
    existValue = False
    for y in {Ann, Bob, Chris}:
        existValue = existValue ∨ L(x,y)
    end
    univValue = existValue ∧ univValue
end
Return univValue
```

Think about boolean logic

Domain: {Ann, Bob, Chris} $\exists y \forall x\, L(x,y)$

$$
\begin{aligned}
&(\ L(Ann, Ann) \land L(Bob, Ann) \land L(Chris, Ann)\ ) \\
\lor\ &(\ L(Ann, Bob) \land L(Bob, Bob) \land L(Chris, Bob)\ ) \\
\lor\ &(\ L(Ann, Chris) \land L(Bob, Chris) \land L(Chris, Chris)\ )
\end{aligned}
$$

Think about boolean logic

Domain: {Ann, Bob, Chris} $\exists y \forall x \, L(x,y)$

How will this change for "$\forall x \exists y \, L(x,y)$"?

$$
\begin{aligned}
& (\; L(Ann, Ann) \wedge L(Bob, Ann) \wedge L(Chris, Ann) \;) \\
\vee \;& (\; L(Ann, Bob) \wedge L(Bob, Bob) \wedge L(Chris, Bob) \;) \\
\vee \;& (\; L(Ann, Chris) \wedge L(Bob, Chris) \wedge L(Chris, Chris) \;)
\end{aligned}
$$

Think about boolean logic

Domain: {Ann, Bob, Chris} $\exists y \forall x \; L(x,y)$

How will this change for "$\forall x \exists y \; L(x,y)$"?

$$
\begin{aligned}
& (\; L(Ann, Ann) \lor L(Ann, Bob) \lor L(Ann, Chris) \;) \\
\land \; & (\; L(Bob, Ann) \lor L(Bob, Bob) \lor L(Bob, Chris) \;) \\
\land \; & (\; L(Chris, Ann) \lor L(Chris, Bob) \lor L(Chris, Chris) \;)
\end{aligned}
$$

Entailment with Quantifiers

Remember, entailment was about just focusing on one of the consequences of knowing that something is true.

For example, if I know that everybody hates Chris, then I know Raymond hates Chris.

Entailment with Quantifiers

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For example, if I know that everybody hates Chris, then I know Raymond hates Chris.

$$\forall x \neg L(x, \text{Chris}) \models \neg L(\text{Raymond}, \text{Chris})$$

The entailed statement doesn't contain as much information as the original statement-- we threw out some info

Entailment with Quantifiers

$$\forall x \in \mathbb{N}.\ P(x) \models P(2102)$$

The entailed statement doesn’t contain as much information as the original statement-- we threw out some info

Entailment with Quantifiers

$$\forall x \in \mathbb{N}.\ P(x)$$
$$\models P(2102)$$
$$\models \exists x \in \mathbb{N}.\ P(x)$$

The entailed statement doesn't contain as much information as the original statement-- we threw out some info

Entailment with Quantifiers

We might want to say:

$\forall x \in S. P(x) \models \exists x \in S. P(x)$

Where S is any set (domain)

Can you think of a counter example?

Entailment with Quantifiers

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$$\forall x \in S. P(x) \models \exists x \in S. P(x)$$

Where S is any set (domain)

Can you think of a counter example?

Proof by Cases

The general structure of ***PROOF by CASES*** is that of a ***disjunctive tautology***

Proof by Cases

The general structure of ***PROOF by CASES*** is that of a ***disjunctive tautology***

Different situations or’d together that evaluate to true:

(case 1) $\vee$ (case 2) $\vee$ (case 3) $\vee$ (case 4)

Proof by Cases

***Theorem:* __________**

***Proof:* Either (case 1) or (case 2) or (case 3)**

Case 1:
Assume case 1 is true
……
∴ x

Case 2:
Assume case 2 is true
……
∴ x

Case 3:
Assume case 3 is true
……
∴ x

***Since* __________ is true in all cases, it is true in general.**

Proof by Cases -- EXAMPLE

Theorem: **$P \rightarrow Q \equiv \neg P \lor Q$**

Proof: **Either P is true or P is false.**

Proof by Cases -- EXAMPLE

Theorem: **$P \rightarrow Q \equiv \neg P \vee Q$**

Proof: **Either P is true or P is false.** **(a disjunctive tautology!)**

Proof by Cases -- EXAMPLE

Theorem: **$P \rightarrow Q \equiv \neg P \lor Q$**

Proof: **Either P is true or P is false.**

Case 1:

P is True

$P \rightarrow Q \equiv T \rightarrow Q \equiv Q$

$\neg P \lor Q \equiv \neg T \lor Q \equiv F \lor Q \equiv Q$

$\therefore$ $P \rightarrow Q \equiv \neg P \lor Q$ when P is True

Proof by Cases -- EXAMPLE

Theorem: **$P \to Q \equiv \neg P \lor Q$**

Proof: **Either P is true or P is false.**

Case 1:

P is True

$P \to Q \equiv T \to Q \equiv Q$

$\neg P \lor Q \equiv \neg T \lor Q \equiv F \lor Q \equiv Q$

$\therefore$ $P \to Q \equiv \neg P \lor Q$ when P is True

Case 2:

P is False

$P \to Q \equiv F \to Q \equiv T$

$\neg P \lor Q \equiv \neg F \lor Q \equiv T \lor Q \equiv T$

$\therefore$ $P \to Q \equiv \neg P \lor Q$ when P is False

Proof by Cases -- EXAMPLE

Theorem: **$P \to Q \equiv \neg P \lor Q$**

Proof: **Either P is true or P is false.**

Case 1:

P is True

$P \to Q \equiv T \to Q \equiv Q$

$\neg P \lor Q \equiv \neg T \lor Q \equiv F \lor Q \equiv Q$

$\therefore$ $P \to Q \equiv \neg P \lor Q$ when P is True

Case 2:

P is False

$P \to Q \equiv F \to Q \equiv T$

$\neg P \lor Q \equiv \neg F \lor Q \equiv T \lor Q \equiv T$

$\therefore$ $P \to Q \equiv \neg P \lor Q$ when P is False

Since $P \to Q \equiv \neg P \lor Q$ is true in all cases, it is true in general.

Proof by Cases -- EXAMPLE

This was an informal proof, since I used symbols, not English. Prose proof version from:
https://www.cs.virginia.edu/luther/2102/F2020/techniques-q4.html#apply-entailment

Example — This is a full proof of one of our known equivalences

Theorem 1. $P \to Q \equiv \neg P \lor Q$

Proof. Either P is true or P is false.

Case 1: P is true

The expression $P \to Q$ in this case is equivalent to $\top \to Q$, which can be simplified to Q.

The expression $\neg P \lor Q$ in this case is equivalent to $\bot \lor Q$, which can be simplified to Q.

Since the two are equivalent to the same thing, they are equivalent to each other.

Case 2: P is false

The expression $P \to Q$ in this case is equivalent to $\bot \to Q$, which can be simplified to $\top$.

The expression $\neg P \lor Q$ in this case is equivalent to $\top \lor Q$, which can be simplified to $\top$.

Since the two are equivalent to the same thing, they are equivalent to each other.

Since $P \to Q \equiv \neg P \lor Q$ is true in both cases, it is true in general. ∎

Proof by Cases -- EXAMPLE

```
f(x):
     if x%2 = 0, return 2x
     Else return (3x+1)
```

***Theorem*: ∀x ∈ ℕ. f(x) returns an even natural number.**

Proof by Cases -- EXAMPLE

```
f(x):
     if x%2 = 0, return 2x
     Else return (3x+1)
```

***Theorem*: ∀x ∈ ℕ. f(x) returns an even natural number.**

Proof: Either x is even or x is odd

Proof by Cases -- EXAMPLE

```
f(x):
     if x%2 = 0, return 2x
     Else return (3x+1)
```

***Theorem*:** **$\forall x \in \mathbb{N}$. f(x) returns an even natural number.**

Proof: Either x is even or x is odd

Case 1: x is even

In this case, we use the “if” branch of the function and return an even natural number multiplied by 2, which is an even natural number. Therefore, the theorem is true if x is even.

Proof by Cases -- EXAMPLE

```
f(x):
     if x%2 = 0, return 2x
     Else return (3x+1)
```

Theorem**:** **∀x ∈ ℕ. f(x) returns an even natural number.**

Proof: Either x is even or x is odd

Case 1: x is even

In this case, we use the “if” branch of the function and return an even natural number multiplied by 2, which is an even natural number. Therefore, the theorem is true if x is even.

Case 2: x is odd

In this case, we use the “else” branch of the function. An odd natural number multiplied by 3 is an odd natural number. Next, we add one to this odd natural number, which results in an even natural number. Therefore, the theorem is true if x is odd.

Since the theorem is true in all cases, it is true in general.

Proof by Cases -- EXAMPLE

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Example — This is a full proof of one of our known equivalences

Theorem 1. $P \to Q \equiv \neg P \vee Q$

Proof. Either P is true or P is false.

Case 1: P is true

The expression $P \to Q$ in this case is equivalent to $\top \to Q$, which can be simplified to Q.

The expression $\neg P \vee Q$ in this case is equivalent to $\bot \vee Q$, which can be simplified to Q.

Since the two are equivalent to the same thing, they are equivalent to each other.

Case 2: P is false

The expression $P \to Q$ in this case is equivalent to $\bot \to Q$, which can be simplified to $\top$.

The expression $\neg P \vee Q$ in this case is equivalent to $\top \vee Q$, which can be simplified to $\top$.

Since the two are equivalent to the same thing, they are equivalent to each other.

Since $P \to Q \equiv \neg P \vee Q$ is true in both cases, it is true in general. ∎