Little Harmonic Labyrinth

The Tortoise and Achilles are spending a day at Coney Island. After buying a couple of cotton candies, they decide to take a ride on the Ferris wheel.

Tortoise: This is my favorite ride. One seems to move so far, and yet in reality one gets nowhere.

Achilles: I can see why it would appeal to you. Are you all strapped in?

Tortoise: Yes, I think I've got this buckle done. Well, here we go. Whee!

Achilles: You certainly are exuberant today.

Tortoise: I have good reason to be. My aunt, who is a fortune-teller, told me that a stroke of Good Fortune would befall me today. So I am tingling with anticipation.

Achilles: Don't tell me you believe in fortune-telling!

Tortoise: No . . . but they say it works even if you don't believe in it.

Achilles: Well, that's fortunate indeed.

Tortoise: Ah, what a view of the beach, the crowd, the ocean, the city . . .

Achilles: Yes, it certainly is splendid. Say, look at that helicopter over there. It seems to be flying our way. In fact it's almost directly above us now.

Tortoise: Strange—there's a cable dangling down from it, which is coming very close to us. It's coming so close we could practically grab it.

Achilles: Look! At the end of the line there's a giant hook, with a note.

(He reaches out and snatches the note. They pass by and are on their way down.)

Tortoise: Can you make out what the note says?

Achilles: Yes—it reads, "Howdy, friends. Grab a hold of the hook next time around, for an Unexpected Surprise."

Tortoise: The note's a little corny but who knows where it might lead. Perhaps it's got something to do with that bit of Good Fortune due me. By all means, let's try it!

Achilles: Let's!

(On the trip up they unbuckle their buckles, and at the crest of the ride, they grab for the giant hook. All of a sudden they are whooshed up by the cable, which quickly reels them skyward into the hovering helicopter. A large strong hand helps them in.)

Voice: Welcome aboard—Suckers.

Achilles: Wh—who are you?

Voice: Allow me to introduce myself. I am Hexachlorophene J. Goodfortune, Kidnapper-At-Large, and Devourer of Tortoises par Excellence, at your service.

Tortoise: Gulp!

Achilles (whispering to his friend): Uh-oh—I think that this "Goodfortune" is not exactly what we'd anticipated. *(To Goodfortune)* Ah—if I may be so bold—where are you spiriting us off to?

Goodfortune: Ho ho! To my all-electric kitchen-in-the-sky, where I will prepare THIS tasty morsel—*(leering at the Tortoise as he says this)*—in a delicious pie-in-the-sky! And make no mistake—it's all just for my gobbling pleasure! Ho ho ho!

Achilles: All I can say is you've got a pretty fiendish laugh.

Goodfortune (laughing fiendishly): Ho ho ho! For that remark, my friend, you will pay dearly. Ho ho!

Achilles: Good grief—I wonder what he means by that!

Goodfortune: Very simple—I've got a Sinister Fate in store for both of you! Just you wait! Ho ho ho! Ho ho ho!

Achilles: Yikes!

Goodfortune: Well, we have arrived. Disembark, my friends, into my fabulous all-electric kitchen-in-the-sky.

(They walk inside.)

Let me show you around, before I prepare your fates. Here is my bedroom. Here is my study. Please wait here for me for a moment. I've got to go sharpen my knives. While you're waiting, help yourselves to some popcorn. Ho ho ho! Tortoise pie! Tortoise pie! My favorite kind of pie! *(Exit.)*

Achilles: Oh, boy—popcorn! I'm going to munch my head off!

Tortoise: Achilles! You just stuffed yourself with cotton candy! Besides, how can you think about food at a time like this?

Achilles: Good gravy—oh, pardon me—I shouldn't use that turn of phrase, should I? I mean in these dire circumstances . . .

Tortoise: I'm afraid our goose is cooked.

Achilles: Say—take a gander at all these books old Goodfortune has in his study. Quite a collection of esoterica: *Birdbrains I Have Known; Chess and Umbrella-Twirling Made Easy; Concerto for Tapdancer and Orchestra* . . . Hmmm.

Tortoise: What's that small volume lying open over there on the desk, next to the dodecahedron and the open drawing pad?

Achilles: This one? Why, its title is *Provocative Adventures of Achilles and the Tortoise Taking Place in Sundry Spots of the Globe.*

Tortoise: A moderately provocative title.

Achilles: Indeed—and the adventure it's opened to looks provocative. It's called "Djinn and Tonic".

Tortoise: Hmm . . . I wonder why. Shall we try reading it? I could take the Tortoise's part, and you could take that of Achilles.

Achilles: I'm game. Here goes nothing . . .

(They begin reading "Djinn and Tonic".)

(Achilles has invited the Tortoise over to see his collection of prints by his favorite artist, M. C. Escher.)

Tortoise: These are wonderful prints, Achilles.

Achilles: I knew you would enjoy seeing them. Do you have any particular favorite?

Tortoise: One of my favorites is *Convex and Concave,* where two internally consistent worlds, when juxtaposed, make a completely inconsistent composite world. Inconsistent worlds are always fun places to visit, but I wouldn't want to live there.

Achilles: What do you mean, "fun to visit"? Inconsistent worlds don't EXIST, so how can you visit one?

Tortoise: I beg your pardon, but weren't we just agreeing that in this Escher picture, an inconsistent world is portrayed?

Achilles: Yes, but that's just a two-dimensional world—a fictitious world—a picture. You can't visit that world.

Tortoise: I have my ways . . .

Achilles: How could you propel yourself into a flat picture-universe?

Tortoise: By drinking a little glass of PUSHING-POTION. That does the trick.

Achilles: What on earth is pushing-potion?

Tortoise: It's a liquid that comes in small ceramic phials, and which, when drunk by someone looking at a picture, "pushes" him right into the world of that picture. People who aren't aware of the powers of pushing-potion often are pretty surprised by the situations they wind up in.

Achilles: Is there no antidote? Once pushed, is one irretrievably lost?

Tortoise: In certain cases, that's not so bad a fate. But there is, in fact, another potion—well, not a potion, actually, but an elixir—no, not an elixir, but a—a—

Tortoise: He probably means "tonic".

Achilles: Tonic?

Tortoise: That's the word I was looking for! "POPPING-TONIC" is what it's called, and if you remember to carry a bottle of it in your right hand as you swallow the pushing-potion, it too will be pushed into the picture; then, whenever you get a hankering to "pop" back out into real life, you need only take a swallow of popping-tonic, and presto! You're back in the real world, exactly where you were before you pushed yourself in.

Achilles: That sounds very interesting. What would happen if you took some popping-tonic without having previously pushed yourself into a picture?

Tortoise: I don't precisely know, Achilles, but I would be rather wary of horsing around with these strange pushing and popping liquids. Once I had a friend, a Weasel, who did precisely what you suggested—and no one has heard from him since.

Achilles: That's unfortunate. Can you also carry along the bottle of pushing-potion with you?

Tortoise: Oh, certainly. Just hold it in your left hand, and it too will get pushed right along with you into the picture you're looking at.

Achilles: What happens if you then find a picture inside the picture which you have already entered, and take another swig of pushing-potion?

Tortoise: Just what you would expect: you wind up inside that picture-in-a-picture.

Achilles: I suppose that you have to pop twice, then, in order to extricate yourself from the nested pictures, and re-emerge back in real life.

Tortoise: That's right. You have to pop once for each push, since a push takes you down inside a picture, and a pop undoes that.

Achilles: You know, this all sounds pretty fishy to me . . . Are you sure you're not just testing the limits of my gullibility?

Tortoise: I swear! Look—here are two phials, right here in my pocket. *(Reaches into his lapel pocket, and pulls out two rather large unlabeled phials, in one of which one can hear a red liquid sloshing around, and in the other of which one can hear a blue liquid sloshing around.)* If you're willing, we can try them. What do you say?

Achilles: Well, I guess, ahm, maybe, ahm . . .

Tortoise: Good! I knew you'd want to try it out. Shall we push ourselves into the world of Escher's *Convex and Concave*?

Achilles: Well, ah, . . .

Tortoise: Then it's decided. Now we've got to remember to take along this flask of tonic, so that we can pop back out. Do you want to take that heavy responsibility, Achilles?

Achilles: If it's all the same to you, I'm a little nervous, and I'd prefer letting you, with your experience, manage the operation.

Tortoise: Very well, then.

(So saying, the Tortoise pours two small portions of pushing-potion. Then he picks up the flask of tonic and grasps it firmly in his right hand, and both he and Achilles lift their glasses to their lips.)

Tortoise: Bottoms up!

(They swallow.)

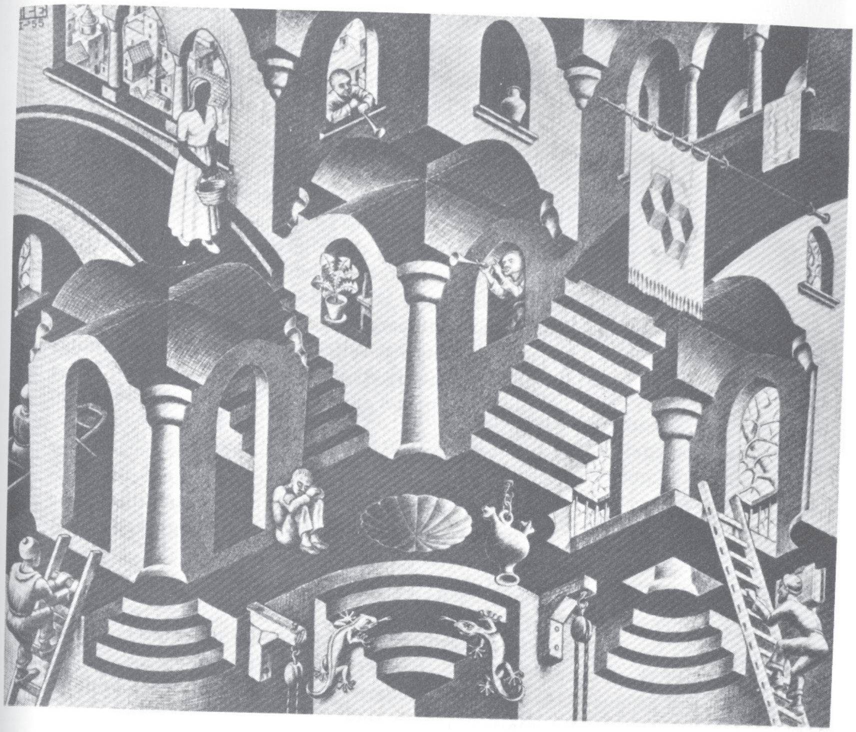

FIGURE 23. Convex and Concave, *by M. C. Escher (lithograph, 1955).*

Achilles: That's an exceedingly strange taste.

Tortoise: One gets used to it.

Achilles: Does taking the tonic feel this strange?

Tortoise: Oh, that's quite another sensation. Whenever you taste the tonic, you feel a deep sense of satisfaction, as if you'd been waiting to taste it all your life.

Achilles: Oh, I'm looking forward to that.

Tortoise: Well, Achilles, where are we?

Achilles (taking cognizance of his surroundings): We're in a little gondola, gliding down a canal! I want to get out. Mr.Gondolier, please let us out here.

(The gondolier pays no attention to this request.)

Tortoise: He doesn't speak English. If we want to get out here, we'd better just clamber out quickly before he enters the sinister "Tunnel of Love", just ahead of us.

(Achilles, his face a little pale, scrambles out in a split second and then pulls his slower friend out.)

Achilles: I didn't like the sound of that place, somehow. I'm glad we got out here. Say, how do you know so much about this place, anyway? Have you been here before?

Tortoise: Many times, although I always came in from other Escher pictures. They're all connected behind the frames, you know. Once you're in one, you can get to any other one.

Achilles: Amazing! Were I not here, seeing these things with my own eyes, I'm not sure I'd believe you. *(They wander out through a little arch.)* Oh, look at those two cute lizards!

Tortoise: Cute? They aren't cute—it makes me shudder just to think of them! They are the vicious guardians of that magic copper lamp hanging from the ceiling over there. A mere touch of their tongues, and any mortal turns to a pickle.

Achilles: Dill, or sweet?

Tortoise: Dill.

Achilles: Oh, what a sour fate! But if the lamp has magical powers, I would like to try for it.

Tortoise: It's a foolhardy venture, my friend. I wouldn't risk it.

Achilles: I'm going to try just once.

(He stealthily approaches the lamp, making sure not to awaken the sleeping lad nearby. But suddenly, he slips on a strange shell-like indentation in the floor, and lunges out into space. Lurching crazily, he reaches for anything, and manages somehow to grab onto the lamp with one hand. Swinging wildly, with both lizards hissing and thrusting their tongues violently out at him, he is left dangling helplessly out in the middle of space.)

Achilles: He-e-e-elp!

(His cry attracts the attention of a woman who rushes downstairs and awakens the sleeping boy. He takes stock of the situation, and, with a kindly smile on his face, gestures to Achilles that all will be well. He shouts something in a strange guttural tongue to a pair of trumpeters high up in windows, and immediately, weird tones begin ringing out and making beats with each other. The sleepy young lad points at the lizards, and Achilles sees that the music is having a strong soporific effect on them. Soon, they are completely unconscious. Then the helpful lad shouts to two companions climbing up ladders. They both pull their ladders up and then extend them out into space just underneath the stranded Achilles, forming a sort of bridge. Their gestures make it clear that Achilles should hurry and climb on. But before he does so, Achilles carefully unlinks the top link of the chain holding the lamp, and detaches the lamp. Then he climbs onto the ladder-bridge and the three young lads pull him in to safety. Achilles throws his arms around them and hugs them gratefully.)

Achilles: Oh, Mr. T, how can I repay them?

Tortoise: I happen to know that these valiant lads just love coffee, and down in the town below, there's a place where they make an incomparable cup of espresso. Invite them for a cup of espresso!

Achilles: That would hit the spot.

(And so, by a rather comical series of gestures, smiles, and words, Achilles manages to convey his invitation to the young lads, and the party of five walks out and down a steep staircase descending into the town. They reach a charming small café, sit down outside, and order five espressos. As they sip their drinks, Achilles remembers he has the lamp with him.)

Achilles: I forgot, Mr. Tortoise—I've got this magic lamp with me! But—what's magic about it?

Tortoise: Oh, you know, just the usual—a genie.

Achilles: What? You mean a genie comes out when you rub it, and grants you wishes?

Tortoise: Right. What did you expect? Pennies from heaven?

Achilles: Well, this is fantastic! I can have any wish I want, eh? I've always wished this would happen to me . . .

(And so Achilles gently rubs the large letter 'L' which is etched on the lamp's copper surface . . . Suddenly a huge puff of smoke appears, and in the forms of the smoke the five friends can make out a weird, ghostly figure towering above them.)

Genie: Hello, my friends—and thanks ever so much for rescuing my Lamp from the evil Lizard-Duo.

(And so saying, the Genie picks up the Lamp, and stuffs it into a pocket concealed among the folds of his long ghostly robe which swirls out of the Lamp.)

As a sign of gratitude for your heroic deed, I would like to offer you, on the part of my Lamp, the opportunity to have any three of your wishes realized.

Achilles: How stupefying! Don't you think so, Mr. T?

Tortoise: I surely do. Go ahead, Achilles, take the first wish.

Achilles: Wow! But what should I wish? Oh, I know! It's what I thought of the first time I read the *Arabian Nights* (that collection of silly (and nested) tales)—I wish that I had a HUNDRED wishes, instead of just three! Pretty clever, eh, Mr. T? I bet YOU never would have thought of that trick. I always wondered why those dopey people in the stories never tried it themselves.

Tortoise: Maybe now you'll find out the answer.

Genie: I am sorry, Achilles, but I don't grant meta-wishes.

Achilles: I wish you'd tell me what a "meta-wish" is!

Genie: But THAT is a meta-meta-wish, Achilles—and I don't grant them, either.

Achilles: Whaaat? I don't follow you at all.

Tortoise: Why don't you rephrase your last request, Achilles?

Achilles: What do you mean? Why should I?

Tortoise: Well, you began by saying "I wish". Since you're just asking for information, why don't you just ask a question?

Achilles: All right, though I don't see why. Tell me, Mr. Genie—what is a meta-wish?

Genie: It is simply a wish about wishes. I am not allowed to grant meta-wishes. It is only within my purview to grant plain ordinary wishes, such as wishing for ten bottles of beer, to have Helen of Troy on a blanket, or to have an all-expenses-paid weekend for two at the Copacabana. You know—simple things like that. But meta-wishes I cannot grant. GOD won't permit me to.

Achilles: GOD? Who is GOD? And why won't he let you grant meta-wishes? That seems like such a puny thing compared to the others you mentioned.

Genie: Well, it's a complicated matter, you see. Why don't you just go ahead and make your three wishes? Or at least make one of them. I don't have all the time in the world, you know . . .

Achilles: Oh, I feel so rotten. I was REALLY HOPING to wish for a hundred wishes . . .

Genie: Gee, I hate to see anybody so disappointed as that. And besides, meta-wishes are my favorite kind of wish. Let me just see if there isn't anything I can do about this. This'll just take one moment—

(The Genie removes from the wispy folds of his robe an object which looks just like the copper Lamp he had put away, except that this one is made of silver; and where the previous one had 'L' etched on it, this one has 'ML' in smaller letters, so as to cover the same area.)

Achilles: And what is that?

Genie: This is my Meta-Lamp . . .

(He rubs the Meta-Lamp, and a huge puff of smoke appears. In the billows of smoke, they can all make out a ghostly form towering above them.)

Meta-Genie: I am the Meta-Genie. You summoned me, O Genie? What is your wish?

Genie: I have a special wish to make of you, O Djinn, and of GOD. I wish for permission for temporary suspension of all type-restrictions on wishes, for the duration of one Typeless Wish. Could you please grant this wish for me?

Meta-Genie: I'll have to send it through Channels, of course. One half a moment, please.

(And, twice as quickly as the Genie did, this Meta-Genie removes from the wispy folds of her robe an object which looks just like the silver Meta-Lamp, except that it is made of gold; and where the previous one had 'ML' etched on it, this one has 'MML' in smaller letters, so as to cover the same area.)

Achilles (his voice an octave higher than before): And what is that?

Meta-Genie: This is my Meta-Meta-Lamp . . .

(She rubs the Meta-Meta-Lamp, and a huge puff of smoke appears. In the billows of smoke, they can all make out a ghostly form towering above them.)

Meta-Meta-Genie: I am the Meta-Meta-Genie. You summoned me, O Meta-Genie? What is your wish?

Meta-Genie: I have a special wish to make of you, O Djinn, and of GOD. I wish for permission for temporary suspension of all type-restrictions on wishes, for the duration of one Typeless Wish. Could you please grant this wish for me?

Meta-Meta-Genie: I'll have to send it through Channels, of course. One quarter of a moment, please.

(And, twice as quickly as the Meta-Genie did, this Meta-Meta-Genie removes from the folds of his robe an object which looks just like the gold Meta-Lamp, except that it is made of . . .)

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. . : : *{GOD}*

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(. . . swirls back into the Meta-Meta-Meta-Lamp, which the Meta-Meta-Genie then folds back into his robe, half as quickly as the Meta-Meta-Meta-Genie did.)

Your wish is granted, O Meta-Genie.

Meta-Genie: Thank you, O Djinn, and GOD.

(And the Meta-Meta-Genie, as all the higher ones before him, swirls back into the Meta-Meta-Lamp, which the Meta-Genie then folds back into her robe, half as quickly as the Meta-Meta-Genie did.)

Your wish is granted, O Genie.

Genie: Thank you, O Djinn, and GOD.

(And the Meta-Genie, as all the higher ones before her, swirls back into the Meta-Lamp, which the Genie then folds back into his robe, half as quickly as the Meta-Genie did.)

Your wish is granted, Achilles.

(And one precise moment has elapsed since he said "This will just take one moment.")

Achilles: Thank you, O Djinn, and GOD.

Genie: I am pleased to report, Achilles, that you may have exactly one (1) Typeless Wish—that is to say, a wish, or a meta-wish, or a meta-meta-wish, as many "meta"'s as you wish—even infinitely many (if you wish).

Achilles: Oh, thank you so very much, Genie. But my curiosity is provoked. Before I make my wish, would you mind telling me who—or what—GOD is?

Genie: Not at all. "GOD" is an acronym which stands for "GOD Over Djinn". The word "Djinn" is used to designate Genies, Meta-Genies, Meta-Meta-Genies, etc. It is a Typeless word.

Achilles: But—but—how can "GOD" be a word in its own acronym? That doesn't make any sense!

Genie: Oh, aren't you acquainted with recursive acronyms? I thought everybody knew about them. You see, "GOD" stands for "GOD Over Djinn"—which can be expanded as "GOD Over Djinn, Over Djinn"—and that can, in turn, be expanded to "GOD Over Djinn, Over Djinn, Over Djinn"—which can, in its turn, be further expanded . . . You can go as far as you like.

Achilles: But I'll never finish!

Genie: Of course not. You can never totally expand GOD.

Achilles: Hmm . . . That's puzzling. What did you mean when you said to the Meta-Genie, "I have a special wish to make of you, O Djinn, and of GOD"?

Genie: I wanted not only to make a request of the Meta-Genie, but also of all the Djinns over her. The recursive acronym method accomplishes this quite naturally. You see, when the Meta-Genie received my request, she then had to pass it upwards to her GOD. So she forwarded a similar message to the Meta-Meta-Genie, who then did likewise to the Meta-Meta-Meta-Genie . . . Ascending the chain this way transmits the message to GOD.

Achilles: I see. You mean GOD sits up at the top of the ladder of djinns?

Genie: No, no, no! There is nothing "at the top", for there is no top. That is why GOD is a recursive acronym. GOD is not some ultimate djinn; GOD is the tower of djinns above any given djinn.

Tortoise: It seems to me that each and every djinn would have a different concept of what GOD is, then, since to any djinn, GOD is the set of djinns above him or her, and no two djinns share that set.

Genie: You're absolutely right—and since I am the lowest djinn of all, my notion of GOD is the most exalted one. I pity the higher djinns, who fancy themselves somehow closer to GOD. What blasphemy!

Achilles: By gum, it must have taken genies to invent GOD.

Tortoise: Do you really believe all this stuff about GOD, Achilles?

Achilles: Why certainly, I do. Are you atheistic, Mr. T? Or are you agnostic?

Tortoise: I don't think I'm agnostic. Maybe I'm meta-agnostic.

Achilles: Whaaat? I don't follow you at all.

Tortoise: Let's see . . . If I were meta-agnostic, I'd be confused over whether I'm agnostic or not—but I'm not quite sure if I feel THAT way; hence I must be meta-meta-agnostic (I guess). Oh, well. Tell me, Genie, does any djinn ever make a mistake, and garble up a message moving up or down the chain?

Genie: This does happen; it is the most common cause for Typeless Wishes not being granted. You see, the chances are infinitesimal that a garbling will occur at any PARTICULAR link in the chain—but when you put an infinite number of them in a row, it becomes virtually certain that a garbling will occur SOMEWHERE. In fact, strange as it seems, an infinite number of garblings usually occur, although they are very sparsely distributed in the chain.

Achilles: Then it seems a miracle that any Typeless Wish ever gets carried out.

Genie: Not really. Most garblings are inconsequential, and many garblings tend to cancel each other out. But occasionally—in fact, rather seldom—the nonfulfillment of a Typeless Wish can be traced back to a single unfortunate djinn's garbling. When this happens, the guilty djinn is forced to run an infinite gauntlet, and get paddled on his or her rump, by GOD. It's good fun for the paddlers, and quite harmless for the paddlee. You might be amused by the sight.

Achilles: I would love to see that! But it only happens when a Typeless Wish goes ungranted?

Genie: That's right.

Achilles: Hmm . . . That gives me an idea for my wish.

Tortoise: Oh, really? What is it?

Achilles: I wish my wish would not be granted!

(At that moment, an event—or is "event" the word for it?—takes place which cannot be described, and hence no attempt will be made to describe it.)

Achilles: What on earth does that cryptic comment mean?

Tortoise: It refers to the Typeless Wish Achilles made.

Achilles: But he hadn't yet made it.

Tortoise: Yes, he had. He said, "I wish my wish would not be granted", and the Genie took THAT to be his wish.

(At that moment, some footsteps are heard coming down the hallway in their direction.)

Achilles: Oh, my! That sounds ominous.

(The footsteps stop; then they turn around and fade away.)

Tortoise: Whew!

Achilles: But does the story go on, or is that the end? Turn the page and let's see.

(The Tortoise turns the page of "Djinn and Tonic", where they find that the story goes on . . .)

Achilles: Hey! What happened? Where is my Genie? My lamp? My cup of espresso? What happened to our young friends from the Convex and Concave worlds? What are all those little lizards doing here?

Tortoise: I'm afraid our context got restored incorrectly, Achilles.

Achilles: What on earth does that cryptic comment mean?

Tortoise: I refer to the Typeless Wish you made.

Achilles: But I hadn't yet made it.

Tortoise: Yes, you had. You said, "I wish my wish would not be granted", and the Genie took THAT to be your wish.

Achilles: Oh, my! That sounds ominous.

Tortoise: It spells PARADOX. For that Typeless Wish to be granted, it had to be denied—yet not to grant it would be to grant it.

Achilles: So what happened? Did the earth come to a standstill? Did the universe cave in?

Tortoise: No. The System crashed.

Achilles: What does that mean?

Tortoise: It means that you and I, Achilles, were suddenly and instantaneously transported to Tumbolia.

Achilles: To where?

Tortoise: Tumbolia: the land of dead hiccups and extinguished light bulbs. It's a sort of waiting room, where dormant software waits for its host hardware to come back up. No telling how long the System was down, and we were in Tumbolia. It could have been moments, hours, days—even years.

Achilles: I don't know what software is, and I don't know what hardware is. But I do know that I didn't get to make my wishes! I want my Genie back!

Tortoise: I'm sorry, Achilles—you blew it. You crashed the System, and you should thank your lucky stars that we're back at all. Things could have come out a lot worse. But I have no idea where we are.

Achilles: I recognize it now—we're inside another of Escher's pictures. This time it's *Reptiles*.

Tortoise: Aha! The System tried to save as much of our context as it could before it crashed, and it got as far as recording that it was an Escher picture with lizards before it went down. That's commendable.

Achilles: And look—isn't that our phial of popping-tonic over there on the table, next to the cycle of lizards?

Tortoise: It certainly is, Achilles. I must say, we are very lucky indeed. The System was very kind to us, in giving us back our popping-tonic—it's precious stuff!

Achilles: I'll say! Now we can pop back out of the Escher world, into my house.

Tortoise: There are a couple of books on the desk, next to the tonic. I wonder what they are. *(He picks up the smaller one, which is open to a random page.)* This looks like a moderately provocative book.

Achilles: Oh, really? What is its title?

Tortoise: *Provocative Adventures of the Tortoise and Achilles Taking Place in Sundry Parts of the Globe.* It sounds like an interesting book to read out of.

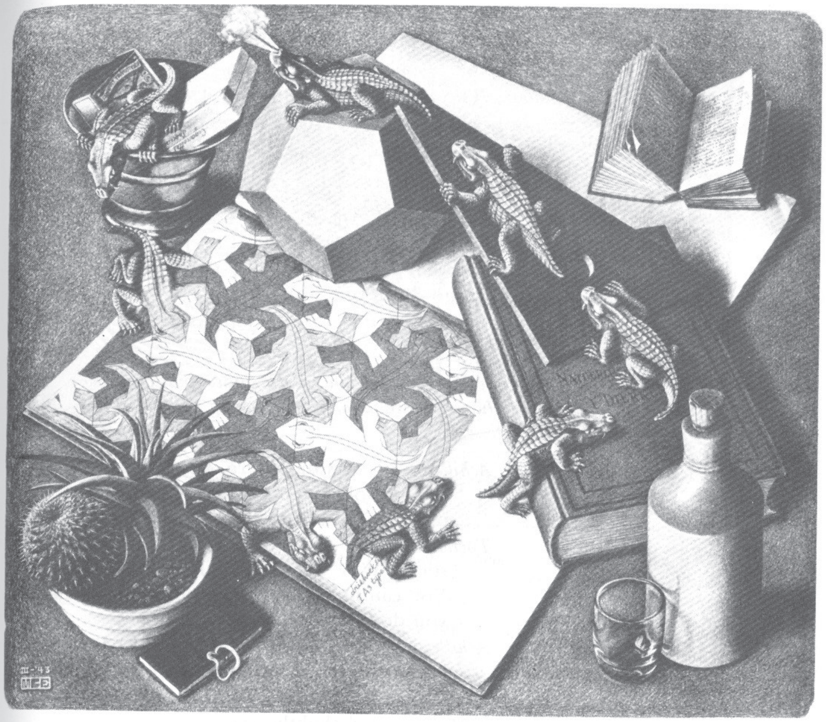


FIGURE 24. Reptiles, *by M. C. Escher (lithograph, 1943).*

Achilles: Well, YOU can read it if you want, but as for me, I'm not going to take any chances with that popping-tonic—one of the lizards might knock it off the table, so I'm going to get it right now!

(He dashes over to the table and reaches for the popping-tonic, but in his haste he somehow bumps the flask of tonic, and it tumbles off the desk and begins rolling.)

Oh, no! Mr. T—look! I accidentally knocked the tonic onto the floor, and it's rolling towards—towards—the stairwell! Quick—before it falls!

(The Tortoise, however, is completely wrapped up in the thin volume which he has in his hands.)

Tortoise (muttering): Eh? This story looks fascinating.

Achilles: Mr. T, Mr. T, help! Help catch the tonic-flask!

Tortoise: What's all the fuss about?

Achilles: The tonic-flask—I knocked it down from the desk, and now it's rolling and—

(At that instant it reaches the brink of the stairwell, and plummets over . . .)

Oh no! What can we do? Mr. Tortoise—aren't you alarmed? We're losing our tonic! It's just fallen down the stairwell! There's only one thing to do! We'll have to go down one story!

Tortoise: Go down one story? My pleasure. Won't you join me?

(He begins to read aloud, and Achilles, pulled in two directions at once, finally stays, taking the role of the Tortoise.)

Achilles: It's very dark here, Mr. T. I can't see a thing. Oof! I bumped into a wall. Watch out!

Tortoise: Here—I have a couple of walking sticks. Why don't you take one of them? You can hold it out in front of you so that you don't bang into things.

Achilles: Good idea. *(He takes the stick.)* Do you get the sense that this path is curving gently to the left as we walk?

Tortoise: Very slightly, yes.

Achilles: I wonder where we are. And whether we'll ever see the light of day again. I wish I'd never listened to you, when you suggested I swallow some of that "DRINK ME" stuff.

Tortoise: I assure you, it's quite harmless. I've done it scads of times, and not a once have I ever regretted it. Relax and enjoy being small.

Achilles: Being small? What is it you've done to me, Mr. T?

Tortoise: Now don't go blaming me. You did it of your own free will.

Achilles: Have you made me shrink? So that this labyrinth we're in is actually some teeny thing that someone could STEP on?

FIGURE 25. Cretan Labyrinth (Italian engraving; School of Finiguerra*).* [*From W. H. Matthews,* Mazes and Labyrinths: Their History and Development *(New York: Dover Publications, 1970).*]

Tortoise: Labyrinth? Labyrinth? Could it be? Are we in the notorious Little Harmonic Labyrinth of the dreaded Majotaur?

Achilles: Yiikes! What is that?

Tortoise: They say—although I personally never believed it myself—that an Evil Majotaur has created a tiny labyrinth and sits in a pit in the middle of it, waiting for innocent victims to get lost in its fearsome complexity. Then, when they wander lost and dazed into the center, he laughs and laughs at them—so hard, that he laughs them to death!

Achilles: Oh, no!

Tortoise: But it's only a myth. Courage, Achilles.

(And the dauntless pair trudge on.)

Achilles: Feel these walls. They're like corrugated tin sheets, or something. But the corrugations have different sizes.

(To emphasize his point, he sticks out his walking stick against the wall surface as he walks. As the stick bounces back and forth against the corrugations, strange noises echo up and down the long curved corridor they are in.)

Tortoise (alarmed): What was THAT?

Achilles: Oh, just me, rubbing my walking stick against the wall.

Tortoise: Whew! I thought for a moment it was the bellowing of the ferocious Majotaur!

Achilles: I thought you said it was all a myth.

Tortoise: Of course it is. Nothing to be afraid of.

(Achilles puts his walking stick back against the wall, and continues walking. As he does so, some musical sounds are heard, coming from the point where his stick is scraping the wall.)

Tortoise: Uh-oh. I have a bad feeling, Achilles. That Labyrinth may not be a myth, after all.

Achilles: Wait a minute. What makes you change your mind all of a sudden?

Tortoise: Do you hear that music?

(To hear more clearly, Achilles lowers the stick, and the strains of melody cease.)

Hey! Put that back! I want to hear the end of this piece!

(Confused, Achilles obeys, and the music resumes.)

Thank you. Now as I was about to say, I have just figured out where we are.

Achilles: Really? Where are we?

Tortoise: We are walking down a spiral groove of a record in its jacket. Your stick scraping against the strange shapes in the wall acts like a needle running down the groove, allowing us to hear the music.

Achilles: Oh, no, oh, no . . .

Tortoise: What? Aren't you overjoyed? Have you ever had the chance to be in such intimate contact with music before?

Achilles: How am I ever going to win footraces against full-sized people when I am smaller than a flea, Mr. Tortoise?

Tortoise: Oh, is that all that's bothering you? That's nothing to fret about, Achilles.

Achilles: The way you talk, I get the impression that you never worry at all.

Tortoise: I don't know. But one thing for certain is that I don't worry about being small. Especially not when faced with the awful danger of the dreaded Majotaur!

Achilles: Horrors! Are you telling me—

Tortoise: I'm afraid so, Achilles. The music gave it away.

Achilles: How could it do that?

Tortoise: Very simple. When I heard the melody B-A-C-H in the top voice, I immediately realized that the grooves that we're walking through could only be the *Little Harmonic Labyrinth,* one of Bach's lesser known organ pieces. It is so named because of its dizzyingly frequent modulations.

Achilles: Wh-what are they?

Tortoise: Well, you know that most musical pieces are written in a key, or tonality, such as C major, which is the key of this one.

Achilles: I had heard the term before. Doesn't that mean that C is the note you want to end on?

Tortoise: Yes, C acts like a home base, in a way. Actually, the usual word is "tonic".

Achilles: Does one then stray away from the tonic with the aim of eventually returning?

Tortoise: That's right. As the piece develops, ambiguous chords and melodies are used, which lead away from the tonic. Little by little, tension builds up—you feel an increasing desire to return home, to hear the tonic.

Achilles: Is that why, at the end of a piece, I always feel so satisfied, as if I had been waiting my whole life to hear the tonic?

Tortoise: Exactly. The composer has used his knowledge of harmonic progressions to manipulate your emotions, and to build up hopes in you to hear that tonic.

Achilles: But you were going to tell me about modulations.

Tortoise: Oh, yes. One very important thing a composer can do is to "modulate" partway through a piece, which means that he sets up a temporary goal other than resolution into the tonic.

Achilles: I see . . . I think. Do you mean that some sequence of chords shifts the harmonic tension somehow so that I actually desire to resolve in a new key?

Tortoise: Right. This makes the situation more complex, for although in the short term you want to resolve in the new key, all the while at the back of your mind you retain the longing to hit that original goal—in this case, C major. And when the subsidiary goal is reached, there is—

Achilles (suddenly gesturing enthusiastically): Oh, listen to the gorgeous upward-swooping chords which mark the end of this *Little Harmonic Labyrinth*!

Tortoise: No, Achilles, this isn't the end. It's merely—

Achilles: Sure it is! Wow! What a powerful, strong ending! What a sense of relief! That's some resolution! Gee!

(And sure enough, at that moment the music stops, as they emerge into an open area with no walls.)

You see, it IS over. What did I tell you?

Tortoise: Something is very wrong. This record is a disgrace to the world of music.

Achilles: What do you mean?

Tortoise: It was exactly what I was telling you about. Here Bach had modulated from C into G, setting up a secondary goal of hearing G. This means that you experience two tensions at once—waiting for resolution into G, but also keeping in mind that ultimate desire—to resolve triumphantly into C Major.

Achilles: Why should you have to keep anything in mind when listening to a piece of music? Is music only an intellectual exercise?

Tortoise: No, of course not. Some music is highly intellectual, but most music is not. And most of the time your ear or brain does the "calculation" for you, and lets your emotions know what they want to hear. You don't have to think about it consciously. But in this piece, Bach was playing tricks, hoping to lead you astray. And in your case, Achilles, he succeeded.

Achilles: Are you telling me that I responded to a resolution in a subsidiary key?

Tortoise: That's right.

Achilles: It still sounded like an ending to me.

Tortoise: Bach intentionally made it sound that way. You just fell into his trap. It was deliberately contrived to sound like an ending, but if you follow the harmonic progression carefully, you will see that it is in the wrong key. Apparently not just you but also this miserable record company fell for the same trick—and they truncated the piece early!

Achilles: What a dirty trick Bach played on me!

Tortoise: That is his whole game—to make you lose your way in his Labyrinth! The Evil Majotaur is in cahoots with Bach, you see. And if you don't watch out, he will now laugh you to death—and perhaps me along with you!

Achilles: Oh, let us hurry up and get out of here! Quick! Let's run backwards in the grooves, and escape on the outside of the record before the Evil Majotaur finds us!

Tortoise: Heavens, no! My sensibility is far too delicate to handle the bizarre chord progressions which occur when time is reversed.

Achilles: Oh, Mr. T, how will we ever get out of here, if we can't just retrace our steps?

Tortoise: That's a very good question.

(A little desperately, Achilles starts running about aimlessly in the dark. Suddenly there is a slight gasp, and then a "thud".)

Achilles—are you all right?

Achilles: Just a bit shaken up but otherwise fine. I fell into some big hole.

Tortoise: You've fallen into the pit of the Evil Majotaur! Here, I'll come help you out. We've got to move fast!

Achilles: Careful, Mr. T—I don't want YOU to fall in here, too . . .

Tortoise: Don't fret, Achilles. Everything will be all—

(Suddenly, there is a slight gasp, and then a "thud".)

Achilles: Mr. T—you fell in, too! Are you all right?

Tortoise: Only my pride is hurt—otherwise I'm fine.

Achilles: Now we're in a pretty pickle, aren't we?

(Suddenly, a giant, booming laugh is heard, alarmingly close to them.)

Tortoise: Watch out, Achilles! This is no laughing matter.

Majotaur: Hee hee hee! Ho ho! Haw haw haw!

Achilles: I'm starting to feel weak, Mr. T . . .

Tortoise: Try to pay no attention to his laugh, Achilles. That's your only hope.

Achilles: I'll do my best. If only my stomach weren't empty!

Tortoise: Say, am I smelling things, or is there a bowl of hot buttered popcorn around here?

Achilles: I smell it, too. Where is it coming from?

Tortoise: Over here, I think. Oh! I just ran into a big bowl of the stuff. Yes, indeed—it seems to be a bowl of popcorn!

Achilles: Oh, boy—popcorn! I'm going to munch my head off!

Tortoise: Let's just hope it isn't pushcorn! Pushcorn and popcorn are so extraordinarily difficult to tell apart.

Achilles: What's this about Pushkin?

Tortoise: I didn't say a thing. You must be hearing things.

Achilles: Go-golly! I hope not. Well, let's dig in!

(And the two friends begin munching the popcorn (or pushcorn?)—and all at once—POP! I guess it was popcorn, after all.)

Tortoise: What an amusing story. Did you enjoy it?

Achilles: Mildly. Only I wonder whether they ever got out of that Evil Majotaur's pit or not. Poor Achilles—he wanted to be full-sized again.

Tortoise: Don't worry—they're out, and he is full-sized again. That's what the "POP" was all about.

Achilles: Oh, I couldn't tell. Well, now I REALLY want to find that bottle of tonic. For some reason, my lips are burning. And nothing would taste better than a drink of popping-tonic.

Tortoise: That stuff is renowned for its thirst quenching powers. Why, in some places people very nearly go crazy over it. At the turn of the century in Vienna, the Schönberg food factory stopped making tonic, and started making cereal instead. You can't imagine the uproar that caused.

Achilles: I have an inkling. But let's go look for the tonic. Hey—just a moment. Those lizards on the desk—do you see anything funny about them?

Tortoise: Umm . . . not particularly. What do you see of such great interest?

Achilles: Don't you see it? They're emerging from that flat picture without drinking any popping-tonic! How are they able to do that?

Tortoise: Oh, didn't I tell you? You can get out of a picture by moving perpendicularly to its plane, if you have no popping-tonic. The little lizards have learned to climb UP when they want to get out of the two-dimensional sketchbook world.

Achilles: Could we do the same thing to get out of this Escher picture we're in?

Tortoise: Of course! We just need to go UP one story. Do you want to try it?

Achilles: Anything to get back to my house! I'm tired of all these provocative adventures.

Tortoise: Follow me, then, up this way.

(And they go up one story.)

Achilles: It's good to be back. But something seems wrong. This isn't my house! This is YOUR house, Mr. Tortoise.

Tortoise: Well, so it is—and am I glad for that! I wasn't looking forward one whit to the long walk back from your house. I am bushed, and doubt if I could have made it.

Achilles: I don't mind walking home, so I guess it's lucky we ended up here, after all.

Tortoise: I'll say! This certainly is a piece of Good Fortune!

CHAPTER V

Recursive Structures and Processes

What Is Recursion?

WHAT IS RECURSION? It is what was illustrated in the Dialogue *Little Harmonic Labyrinth:* nesting, and variations on nesting. The concept is very general. (Stories inside stories, movies inside movies, paintings inside paintings, Russian dolls inside Russian dolls (even parenthetical comments inside parenthetical comments!)—these are just a few of the charms of recursion.) However, you should be aware that the meaning of "recursive" in this Chapter is only faintly related to its meaning in Chapter III. The relation should be clear by the end of this Chapter.

Sometimes recursion seems to brush paradox very closely. For example, there are *recursive definitions.* Such a definition may give the casual viewer the impression that something is being defined in terms of *itself.* That would be circular and lead to infinite regress, if not to paradox proper. Actually, a recursive definition (when properly formulated) never leads to infinite regress or paradox. This is because a recursive definition never defines something in terms of itself, but always in terms of *simpler versions* of itself. What I mean by this will become clearer shortly, when I show some examples of recursive definitions.

One of the most common ways in which recursion appears in daily life is when you postpone completing a task in favor of a simpler task, often of the same type. Here is a good example. An executive has a fancy telephone and receives many calls on it. He is talking to A when B calls. To A he says, "Would you mind holding for a moment?" Of course he doesn't really care if A minds; he just pushes a button, and switches to B. Now C calls. The same deferment happens to B. This could go on indefinitely, but let us not get too bogged down in our enthusiasm. So let's say the call with C terminates. Then our executive "pops" back up to B, and continues. Meanwhile, A is sitting at the other end of the line, drumming his fingernails against some table, and listening to some horrible Muzak piped through the phone lines to placate him . . . Now the easiest case is if the call with B simply terminates, and the executive returns to A finally. But it *could* happen that after the conversation with B is resumed, a new caller—D—calls. B is once again pushed onto the stack of waiting callers, and D is taken care of. After D is done, back to B, then back to A. This executive is hopelessly mechanical, to be sure—but we are illustrating recursion in its most precise form.

Pushing, Popping, and Stacks

In the preceding example, I have introduced some basic terminology of recursion—at least as seen through the eyes of computer scientists. The terms are *push, pop,* and *stack* (or *push-down stack,* to be precise) and they are all related. They were introduced in the late 1950's as part of IPL, one of the first languages for Artificial Intelligence. You have already encountered "push" and "pop" in the Dialogue. But I will spell things out anyway. To *push* means to suspend operations on the task you're currently working on, without forgetting where you are—and to take up a new task. The new task is usually said to be "on a lower level" than the earlier task. To *pop* is the reverse—it means to close operations on one level, and to resume operations exactly where you left off, one level higher.

But how do you remember exactly where you were on each different level? The answer is, you store the relevant information in a *stack.* So a stack is just a table telling you such things as (1) where you were in each unfinished task (jargon: the "return address"), (2) what the relevant facts to know were at the points of interruption (jargon: the "variable bindings"). When you pop back up to resume some task, it is the stack which restores your context, so you don't feel lost. In the telephone-call example, the stack tells you *who* is waiting on each different level, and *where* you were in the conversation when it was interrupted.

By the way, the terms "push", "pop", and "stack" all come from the visual image of cafeteria trays in a stack. There is usually some sort of spring underneath which tends to keep the topmost tray at a constant height, more or less. So when you push a tray onto the stack, it sinks a little—and when you remove a tray from the stack, the stack pops up a little.

One more example from daily life. When you listen to a news report on the radio, oftentimes it happens that they switch you to some foreign correspondent. "We now switch you to Sally Swumpley in Peafog, England." Now Sally has got a tape of some local reporter interviewing someone, so after giving a bit of background, she plays it. "I'm Nigel Cadwallader, here on scene just outside of Peafog, where the great robbery took place, and I'm talking with . . ." Now you are three levels down. It may turn out that the interviewee also plays a tape of some conversation. It is not too uncommon to go down three levels in real news reports, and surprisingly enough, we scarcely have any awareness of the suspension. It is all kept track of quite easily by our subconscious mind. Probably the reason it is so easy is that each level is extremely different in flavor from each other level. If they were all similar, we would get confused in no time flat.

An example of a more complex recursion is, of course, our Dialogue. There, Achilles and the Tortoise appeared on all the different levels. Sometimes they were reading a story in which they appeared as characters. That is when your mind may get a little hazy on what's going on, and you have to concentrate carefully to get things straight. "Let's see, the *real* Achilles and Tortoise are still up there in Goodfortune's helicopter, but the *secondary* ones are in some Escher picture—and then they found this book and are reading in it, so it's the *tertiary* Achilles and Tortoise who are wandering around inside the grooves of the *Little Harmonic Labyrinth.* No, wait a minute—I left out one level somewhere . . ." You have to have a conscious mental stack like this in order to keep track of the recursion in the Dialogue. (See Fig. 26.)

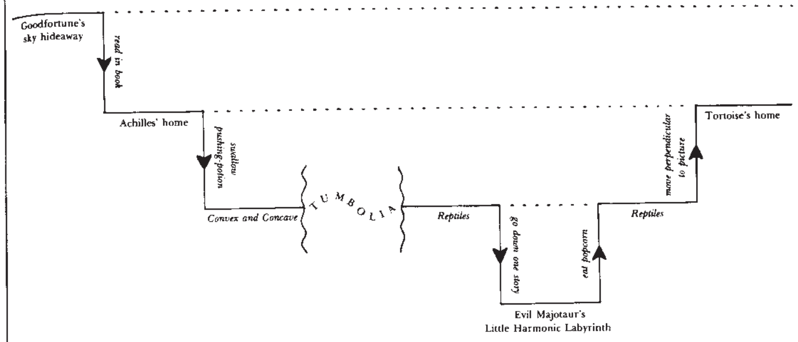


FIGURE 26. Diagram of the structure of the Dialogue Little Harmonic Labyrinth. *Vertical descents are "pushes"; rises are "pops". Notice the similarity of this diagram to the indentation pattern of the Dialogue. From the diagram it is clear that the initial tension—Goodfortune's threat—never was resolved; Achilles and the Tortoise were just left dangling in the sky. Some readers might agonize over this unpopped push, while others might not bat an eyelash. In the story, Bach's musical labyrinth likewise was cut off too soon—but Achilles didn't even notice anything funny. Only the Tortoise was aware of the more global dangling tension.*

Stacks in Music

While we're talking about the *Little Harmonic Labyrinth,* we should discuss something which is hinted at, if not stated explicitly in the Dialogue: that we hear music recursively—in particular, that we maintain a mental stack of keys, and that each new modulation pushes a new key onto the stack. The implication is further that we want to hear that sequence of keys retraced in reverse order—popping the pushed keys off the stack, one by one, until the tonic is reached. This is an exaggeration. There is a grain of truth to it, however.

Any reasonably musical person automatically maintains a shallow stack with two keys. In that "short stack", the true tonic key is held, and also the most immediate "pseudotonic" (the key the composer is pretending to be in). In other words, the most global key and the most local key. That way, the listener knows when the true tonic is regained, and feels a strong sense of "relief". The listener can also distinguish (unlike Achilles) between a *local* easing of tension—for example a resolution into the pseudotonic—

and a *global* resolution. In fact, a pseudoresolution should heighten the global tension, not relieve it, because it is a piece of irony—just like Achilles' rescue from his perilous perch on the swinging lamp, when all the while you know he and the Tortoise are really awaiting their dire fates at the knife of Monsieur Goodfortune.

Since tension and resolution are the heart and soul of music, there are many, many examples. But let us just look at a couple in Bach. Bach wrote many pieces in an *"AABB"* form—that is, where there are two halves, and each one is repeated. Let's take the gigue from the French Suite no. 5, which is quite typical of the form. Its tonic key is G, and we hear a gay dancing melody which establishes the key of G strongly. Soon, however, a modulation in the *A*-section leads to the closely related key of D (the dominant). When the *A*-section ends, we are in the key of D. In fact, it sounds as if the piece has ended in the key of D! (Or at least it might sound that way to Achilles.) But then a strange thing happens—we abruptly jump back to the beginning, back to G, and rehear the same transition into D. But then a strange thing happens—we abruptly jump back to the beginning, back to G, and rehear the same transition into D.

Then comes the *B*-section. With the inversion of the theme for our melody, we begin in D as if that had always been the tonic—but we modulate back to G after all, which means that we pop back into the tonic, and the *B*-section ends properly. Then that funny repetition takes place, jerking us without warning back into D, and letting us return to G once more. Then that funny repetition takes place, jerking us without warning back into D, and letting us return to G once more.

The psychological effect of all this key shifting—some jerky, some smooth—is very difficult to describe. It is part of the magic of music that we can automatically make sense of these shifts. Or perhaps it is the magic of Bach that he can write pieces with this kind of structure which have such a natural grace to them that we are not aware of exactly what is happening.

The original *Little Harmonic Labyrinth* is a piece by Bach in which he tries to lose you in a labyrinth of quick key changes. Pretty soon you are so disoriented that you don't have any sense of direction left—you don't know where the true tonic is, unless you have perfect pitch, or like Theseus, have a friend like Ariadne who gives you a thread that allows you to retrace your steps. In this case, the thread would be a written score. This piece—another example is the Endlessly Rising Canon—goes to show that, as music listeners, we don't have very reliable deep stacks.

Recursion in Language

Our mental stacking power is perhaps slightly stronger in language. The grammatical structure of all languages involves setting up quite elaborate push-down stacks, though, to be sure, the difficulty of understanding a sentence increases sharply with the number of pushes onto the stack. The proverbial German phenomenon of the "verb-at-the-end", about which droll tales of absentminded professors who would begin a sentence, ramble on for an entire lecture, and then finish up by rattling off a string of verbs by which their audience, for whom the stack had long since lost its coherence, would be totally nonplussed, are told, is an excellent example of linguistic pushing and popping. The confusion among the audience that out-of-order popping from the stack onto which the professor's verbs had been pushed, is amusing to imagine, could engender. But in normal spoken German, such deep stacks almost never occur—in fact, native speakers of German often unconsciously violate certain conventions which force the verb to go to the end, in order to avoid the mental effort of keeping track of the stack. Every language has constructions which involve stacks, though usually of a less spectacular nature than German. But there are always ways of rephrasing sentences so that the depth of stacking is minimal.

Recursive Transition Networks

The syntactical structure of sentences affords a good place to present a way of describing recursive structures and processes: the *Recursive Transition Network* (RTN). An RTN is a diagram showing various paths which can be followed to accomplish a particular task. Each path consists of a number of *nodes,* or little boxes with words in them, joined by *arcs,* or lines with arrows. The overall name for the RTN is written separately at the left, and the first and last nodes have the words *begin* and *end* in them. All the other nodes contain either very short explicit directions to perform, or else names of other RTN's. Each time you hit a node, you are to carry out the directions inside it, or to jump to the RTN named inside it, and carry it out.

Let's take a sample RTN, called **ORNATE NOUN**, which tells how to construct a certain type of English noun phrase. (See Fig. 27a.) If we traverse **ORNATE NOUN** purely horizontally, we *begin*, then we create an **ARTICLE**, an **ADJECTIVE**, and a **NOUN**, then we *end*. For instance, "the silly shampoo" or "a thankless brunch". But the arcs show other possibilities, such as skipping the article, or repeating the adjective. Thus we could construct "milk", or "big red blue green sneezes", etc.

When you hit the node **NOUN**, you are asking the unknown black box called **NOUN** to fetch any noun for you from its storehouse of nouns. This is known as a *procedure call,* in computer science terminology. It means you temporarily give control to a *procedure* (here, **NOUN**) which (1) does its thing (produces a noun) and then (2) hands control back to you. In the above RTN, there are calls on three such procedures: **ARTICLE**, **ADJECTIVE**, and **NOUN**. Now the RTN **ORNATE NOUN** could itself be called from some other RTN—for instance an RTN called **SENTENCE**. In this case, **ORNATE NOUN** would produce a phrase such as "the silly shampoo" and then return to the place inside **SENTENCE** from which it had been called. It is quite reminiscent of the way in which you resume where you left off in nested telephone calls or nested news reports.

However, despite calling this a "recursive transition network", we have

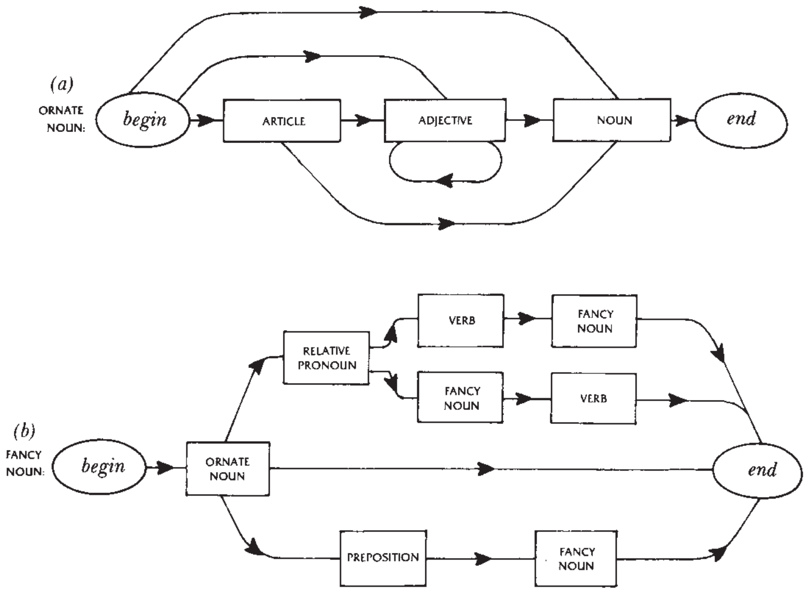


FIGURE 27. Recursive Transition Networks for **ORNATE NOUN** *and* **FANCY NOUN**.

not exhibited any true recursion so far. Things get recursive—and seemingly circular—when you go to an RTN such as the one in Figure 27b, for **FANCY NOUN**. As you can see, every possible pathway in **FANCY NOUN** involves a call on **ORNATE NOUN**, so there is no way to avoid getting a noun of some sort or other. And it is possible to be no more ornate than that, coming out merely with "milk" or "big red blue green sneezes". But three of the pathways involve *recursive* calls on **FANCY NOUN** itself. It certainly looks as if something is being defined in terms of itself. Is that what is happening, or not?

The answer is "yes, but benignly". Suppose that, in the procedure **SENTENCE**, there is a node which calls **FANCY NOUN**, and we hit that node. This means that we commit to memory (viz., the stack) the location of that node inside **SENTENCE**, so we'll know where to return to—then we transfer our attention to the procedure **FANCY NOUN**. Now we must choose a pathway to take, in order to generate a **FANCY NOUN**. Suppose we choose the lower of the upper pathways—the one whose calling sequence goes:

ORNATE NOUN; RELATIVE PRONOUN; FANCY NOUN; VERB.

So we spit out an **ORNATE NOUN**: *"the strange bagels"*; a **RELATIVE PRONOUN**: *"that"*; and now we are suddenly asked for a **FANCY NOUN**. But we are in the middle of **FANCY NOUN**! Yes, but remember our executive who was in the middle of one phone call when he got another one. He merely stored the old phone call's status on a stack, and began the new one as if nothing were unusual. So we shall do the same.

We first write down in our stack the node we are at in the outer call on **FANCY NOUN**, so that we have a "return address"; then we jump to the beginning of **FANCY NOUN** as if nothing were unusual. Now we have to choose a pathway again. For variety's sake, let's choose the lower pathway: **ORNATE NOUN; PREPOSITION; FANCY NOUN**. That means we produce an **ORNATE NOUN** (say *"the purple cow"*), then a **PREPOSITION** (say *"without"*), and once again, we hit the recursion. So we hang onto our hats, and descend one more level. To avoid complexity, let's assume that this time, the pathway we take is the direct one—just **ORNATE NOUN**. For example, we might get *"horns"*. We hit the node **END** in this call on **FANCY NOUN**, which amounts to popping out, and so we go to our stack to find the return address. It tells us that we were in the middle of executing **FANCY NOUN** one level up—and so we resume there. This yields *"the purple cow without horns"*. On this level, too, we hit **END**, and so we pop up once more, this time finding ourselves in need of a **VERB**—so let's choose *"gobbled"*. This ends the highest-level call on **FANCY NOUN**, with the result that the phrase

"the strange bagels that the purple cow without horns gobbled"

will get passed upwards to the patient **SENTENCE**, as we pop for the last time.

As you see, we didn't get into any infinite regress. The reason is that at least one pathway inside the RTN **FANCY NOUN** does *not* involve any recursive calls on **FANCY NOUN** itself. Of course, we could have perversely insisted on always choosing the bottom pathway inside **FANCY NOUN**, and then we would never have gotten finished, just as the acronym "GOD" never got fully expanded. But if the pathways are chosen at random, then an infinite regress of that sort will not happen.

"Bottoming Out" and Heterarchies

This is the crucial fact which distinguishes recursive definitions from circular ones. There is always some part of the definition which avoids self-reference, so that the action of constructing an object which satisfies the definition will eventually "bottom out".

Now there are more oblique ways of achieving recursivity in RTN's than by self-calling. There is the analogue of Escher's *Drawing Hands* (Fig. 135), where each of two procedures calls the other, but not itself. For example, we could have an RTN named **CLAUSE**, which calls **FANCY NOUN** whenever it needs an object for a transitive verb, and conversely, the upper path of **FANCY NOUN** could call **RELATIVE PRONOUN** and then **CLAUSE** whenever it wants a relative clause. This is an example of *indirect* recursion. It is reminiscent also of the two-step version of the Epimenides paradox.

Needless to say, there can be a trio of procedures which call one another, cyclically—and so on. There can be a whole family of RTN's which are all tangled up, calling each other and themselves like crazy. A program which has such a structure in which there is no single "highest level", or "monitor", is called a *heterarchy* (as distinguished from a hierarchy). The term is due, I believe, to Warren McCulloch, one of the first cyberneticists, and a reverent student of brains and minds.

Expanding Nodes

One graphic way of thinking about RTN's is this. Whenever you are moving along some pathway and you hit a node which calls on an RTN, you "expand" that node, which means to replace it by a very small copy of the RTN it calls (see Fig. 28). Then you proceed into the very small RTN!

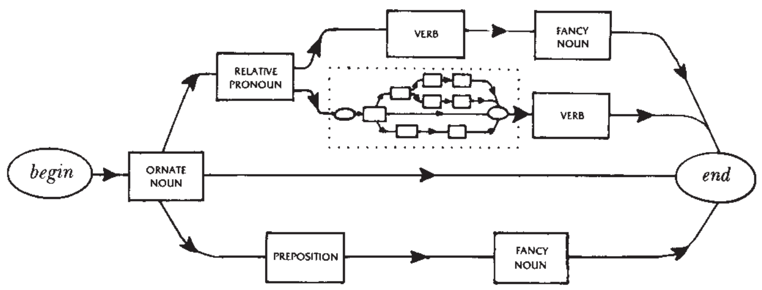


FIGURE 28. The FANCY NOUN *RTN with one node recursively expanded.*

When you pop out of it, you are automatically in the right place in the big one. While in the small one, you may wind up constructing even more miniature RTN's. But by expanding nodes only when you come across them, you avoid the need to make an infinite diagram, even when an RTN calls itself.

Expanding a node is a little like replacing a letter in an acronym by the word it stands for. The "GOD" acronym is recursive but has the defect—or advantage—that you must repeatedly expand the 'G'; thus it never bottoms out. When an RTN is implemented as a real computer program, however, it always has at least one pathway which avoids recursivity (direct or indirect) so that infinite regress is not created. Even the most heterarchical program structure bottoms out—otherwise it couldn't run! It would just be constantly expanding node after node, but never performing any action.

Diagram G and Recursive Sequences

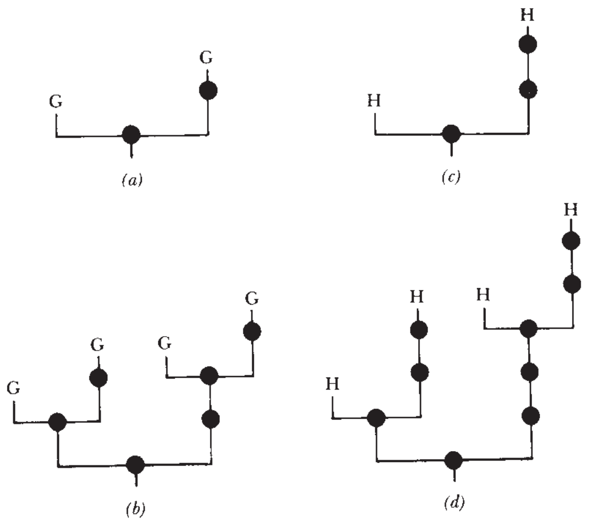
Infinite geometrical structures can be defined in just this way—that is, by expanding node after node. For example, let us define an infinite diagram called "Diagram G". To do so, we shall use an implicit representation. In two nodes, we shall write merely the letter 'G', which, however, will stand for an entire copy of Diagram G. In Figure 29a, Diagram G is portrayed implicitly. Now if we wish to see Diagram G more explicitly, we expand each of the two G's—that is, we *replace them by the same diagram,* only reduced in scale (see Fig. 29b). This "second-order" version of Diagram G gives us an inkling of what the final, impossible-to-realize Diagram G really looks like. In Figure 30 is shown a larger portion of Diagram G, where all the nodes have been numbered from the bottom up, and from left to right. Two extra nodes—numbers 1 and 2—have been inserted at the bottom.

This infinite *tree* has some very curious mathematical properties. Running up its right-hand edge is the famous sequence of *Fibonacci numbers:*

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, . . .

discovered around the year 1202 by Leonardo of Pisa, son of Bonaccio, ergo "Filius Bonacci", or "Fibonacci" for short. These numbers are best

FIGURE 29. (a) Diagram G, unexpanded. (c) Diagram H, unexpanded.
(b) Diagram G, expanded once. (d) Diagram H, expanded once.



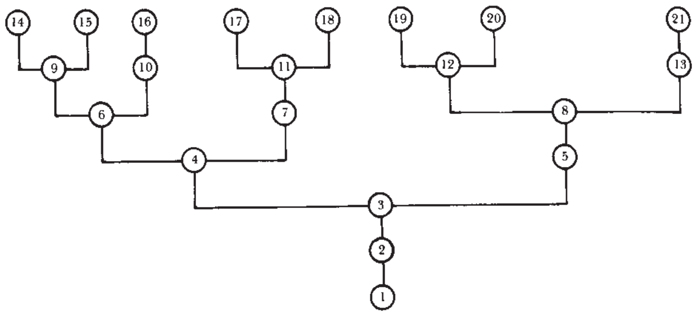


FIGURE 30. Diagram G, further expanded and with numbered nodes.

defined recursively by the pair of formulas

$$\mathrm{FIBO}(n) = \mathrm{FIBO}(n-1) + \mathrm{FIBO}(n-2) \qquad \text{for } n > 2$$
$$\mathrm{FIBO}(1) = \mathrm{FIBO}(2) = 1$$

Notice how new Fibonacci numbers are defined in terms of previous Fibonacci numbers. We could represent this pair of formulas in an RTN (see Fig. 31).

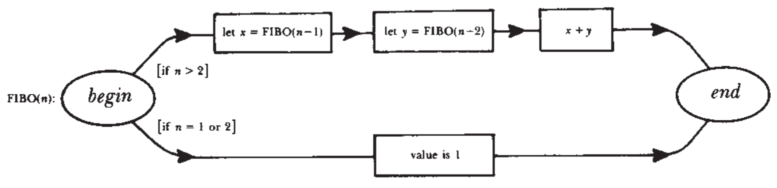


FIGURE 31. An RTN for Fibonacci numbers.

Thus you can calculate FIBO(15) by a sequence of recursive calls on the procedure defined by the RTN above. This recursive definition bottoms out when you hit FIBO(1) or FIBO(2) (which are given explicitly) after you have worked your way backwards through descending values of n. It is slightly awkward to work your way backwards, when you could just as well work your way forwards, starting with FIBO(1) and FIBO(2) and always adding the most recent two values, until you reach FIBO(15). That way you don't need to keep track of a stack.

Now Diagram G has some even more surprising properties than this. Its entire structure can be coded up in a single recursive definition, as follows:

$$\mathrm{G}(n) = n - \mathrm{G}(\mathrm{G}(n-1)) \qquad \text{for } n > 0$$
$$\mathrm{G}(0) = 0$$

How does this function G(n) code for the tree-structure? Quite simply, if you construct a tree by placing G(n) below n, for all values of n, you will recreate Diagram G. In fact, that is how I discovered Diagram G in the first place. I was investigating the *function* G, and in trying to calculate its values quickly, I conceived of displaying the values I already knew in a tree. To my surprise, the tree turned out to have this extremely orderly recursive geometrical description.

What is more wonderful is that if you make the analogous tree for a function H(n) defined with one more nesting than G—

$$\mathrm{H}(n) = n - \mathrm{H}(\mathrm{H}(\mathrm{H}(n-1))) \qquad \text{for } n > 0$$
$$\mathrm{H}(0) = 0$$

—then the associated "Diagram H" is defined implicitly as shown in Figure 29c. The right-hand trunk contains one more node; that is the only difference. The first recursive expansion of Diagram H is shown in Figure 29d. And so it goes, for any degree of nesting. There is a beautiful regularity to the recursive geometrical structures, which corresponds precisely to the recursive algebraic definitions.

A problem for curious readers is: suppose you flip Diagram G around as if in a mirror, and label the nodes of the new tree so they increase from left to right. Can you find a recursive *algebraic* definition for this "flip-tree"? What about for the "flip" of the H-tree? Etc.?

Another pleasing problem involves a pair of recursively intertwined functions F(n) and M(n)—"married" functions, you might say—defined this way:

$$\left.\begin{aligned} \mathrm{F}(n) &= n - \mathrm{M}(\mathrm{F}(n-1)) \\ \mathrm{M}(n) &= n - \mathrm{F}(\mathrm{M}(n-1)) \end{aligned}\right\} \text{ for } n > 0$$
$$\mathrm{F}(0) = 1, \quad \text{and} \quad \mathrm{M}(0) = 0.$$

The RTN's for these two functions call each other and themselves as well. The problem is simply to discover the recursive structures of Diagram F and Diagram M. They are quite elegant and simple.

A Chaotic Sequence

One last example of recursion in number theory leads to a small mystery. Consider the following recursive definition of a function:

$$\mathrm{Q}(n) = \mathrm{Q}(n - \mathrm{Q}(n-1)) + \mathrm{Q}(n - \mathrm{Q}(n-2)) \qquad \text{for } n > 2$$
$$\mathrm{Q}(1) = \mathrm{Q}(2) = 1.$$

It is reminiscent of the Fibonacci definition in that each new value is a sum of two previous values—but not of the *immediately* previous two values. Instead, the two immediately previous values tell *how far to count back* to obtain the numbers to be added to make the new value! The first 17 Q-numbers run as follows:

$$1,\ 1,\ 2,\ 3,\ 3,\ 4,\ 5,\ \underset{\uparrow}{5},\ \underset{\uparrow}{6},\ 6,\ 6,\ 8,\ 8,\ 8,\ \underbrace{10,\ 9,}_{\textit{how far to move to the left}}\ 10, \ldots$$

$$\underbrace{5 + 6 = 11}_{\textit{new term}}$$

To obtain the next one, move leftwards (from the three dots) respectively 10 and 9 terms; you will hit a 5 and a 6, shown by the arrows. Their sum—11—yields the new value: Q(18). This is the strange process by which the list of known Q-numbers is used to extend itself. The resulting sequence is, to put it mildly, erratic. The further out you go, the less sense it seems to make. This is one of those very peculiar cases where what seems to be a somewhat natural definition leads to extremely puzzling behavior: chaos produced in a very orderly manner. One is naturally led to wonder whether the apparent chaos conceals some subtle regularity. Of course, by definition, there is regularity, but what is of interest is whether there is another way of characterizing this sequence—and with luck, a nonrecursive way.

Two Striking Recursive Graphs

The marvels of recursion in mathematics are innumerable, and it is not my purpose to present them all. However, there are a couple of particularly striking examples from my own experience which I feel are worth presenting. They are both graphs. One came up in the course of some number-theoretical investigations. The other came up in the course of my Ph.D. thesis work, in solid state physics. What is truly fascinating is that the graphs are closely related.

The first one (Fig. 32) is a graph of a function which I call INT(x). It is plotted here for x between 0 and 1. For x between any other pair of integers n and $n + 1$, you just find INT($x - n$), then add n back. The structure of the plot is quite jumpy, as you can see. It consists of an infinite number of curved pieces, which get smaller and smaller towards the corners—and incidentally, less and less curved. Now if you look closely at each such piece, you will find that it is actually a copy of the full graph, merely curved! The implications are wild. One of them is that the graph of INT consists of nothing but copies of itself, nested down infinitely deeply. If you pick up any piece of the graph, no matter how small, you are holding a complete copy of the whole graph—in fact, infinitely many copies of it!

The fact that INT consists of nothing but copies of itself might make you think it is too ephemeral to exist. Its definition sounds too circular.

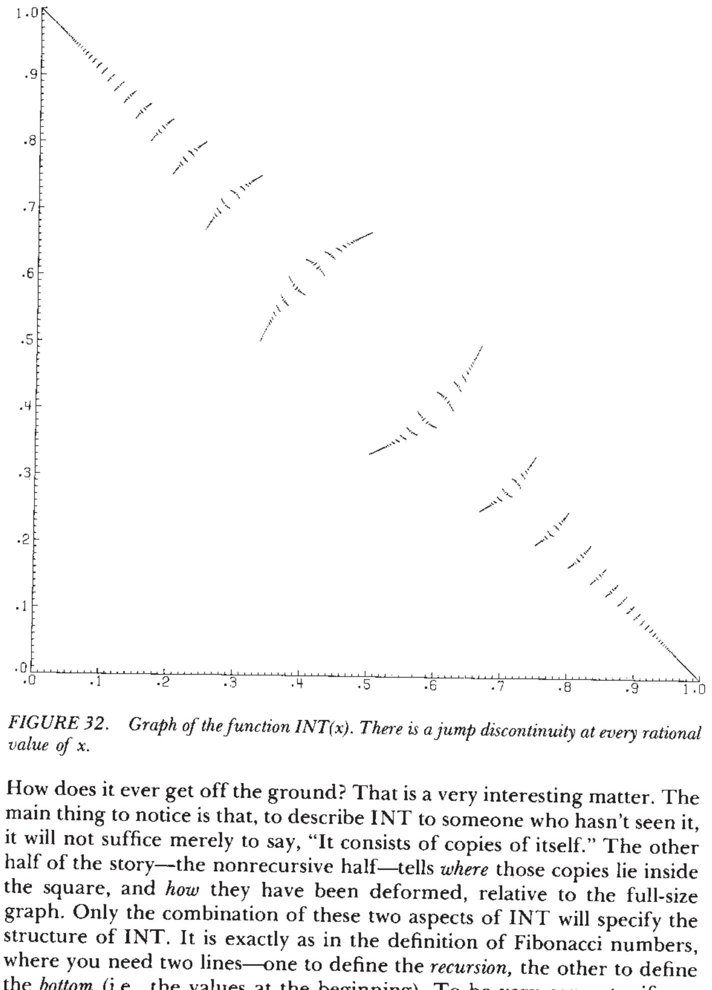


FIGURE 32. Graph of the function INT(x). There is a jump discontinuity at every rational value of x.

How does it ever get off the ground? That is a very interesting matter. The main thing to notice is that, to describe INT to someone who hasn't seen it, it will not suffice merely to say, "It consists of copies of itself." The other half of the story—the nonrecursive half—tells *where* those copies lie inside the square, and *how* they have been deformed, relative to the full-size graph. Only the combination of these two aspects of INT will specify the structure of INT. It is exactly as in the definition of Fibonacci numbers, where you need two lines—one to define the *recursion,* the other to define the *bottom* (i.e., the values at the beginning). To be very concrete, if you make one of the bottom values 3 instead of 1, you will produce a completely different sequence, known as the *Lucas sequence:*

$$\underbrace{1,\ 3,}_{\textit{the ``bottom''}}\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123, \ldots$$

$$29 + 47 = 76$$

same recursive rule
as for the Fibonacci numbers

What corresponds to the *bottom* in the definition of INT is a picture (Fig. 33a) composed of many boxes, showing *where* the copies go, and *how* they are distorted. I call it the "skeleton" of INT. To construct INT from its skeleton, you do the following. First, for each box of the skeleton, you do two operations: (1) put a small curved copy of the skeleton inside the box, using the curved line inside it as a guide; (2) erase the containing box and its curved line. Once this has been done for each box of the original skeleton, you are left with many "baby" skeletons in place of one big one. Next you repeat the process one level down, with all the baby skeletons. Then again, again, and again . . . What you approach in the limit is an exact graph of INT, though you never get there. By nesting the skeleton inside itself over and over again, you gradually construct the graph of INT "from out of nothing". But in fact the "nothing" was not nothing—it was a picture.

To see this even more dramatically, imagine keeping the recursive part of the definition of INT, but changing the initial picture, the skeleton. A variant skeleton is shown in Figure 33b, again with boxes which get smaller and smaller as they trail off to the four corners. If you nest this second skeleton inside itself over and over again, you will create the key graph from my Ph.D. thesis, which I call *Gplot* (Fig. 34). (In fact, some complicated distortion of each copy is needed as well—but nesting is the basic idea.) Gplot is thus a member of the INT-family. It is a distant relative, because its skeleton is quite different from—and considerably more complex than—that of INT. However, the recursive part of the definition is identical, and therein lies the family tie.

I should not keep you too much in the dark about the origin of these beautiful graphs. INT—standing for "interchange"—comes from a problem involving "Eta-sequences", which are related to continued fractions. The basic idea behind INT is that plus and minus signs are interchanged in a certain kind of continued fraction. As a consequence, $\mathrm{INT}(\mathrm{INT}(x)) = x$. INT has the property that if x is rational, so is $\mathrm{INT}(x)$; if x is quadratic, so is $\mathrm{INT}(x)$. I do not know if this trend holds for higher algebraic degrees. Another lovely feature of INT is that at all rational values of x, it has a jump discontinuity, but at all irrational values of x, it is continuous.

Gplot comes from a highly idealized version of the question, "What are the allowed energies of electrons in a crystal in a magnetic field?" This problem is interesting because it is a cross between two very simple and fundamental physical situations: an electron in a perfect crystal, and an electron in a homogeneous magnetic field. These two simpler problems are both well understood, and their characteristic solutions seem almost incompatible with each other. Therefore, it is of quite some interest to see how nature manages to reconcile the two. As it happens, the crystal-without-magnetic-field situation and the magnetic-field-without-crystal situation do have one feature in common: in each of them, the electron behaves periodically in time. It turns out that when the two situations are combined, the ratio of their two time periods is the key parameter. In fact, that ratio holds all the information about the distribution of allowed electron energies—but it only gives up its secret upon being expanded into a continued fraction.

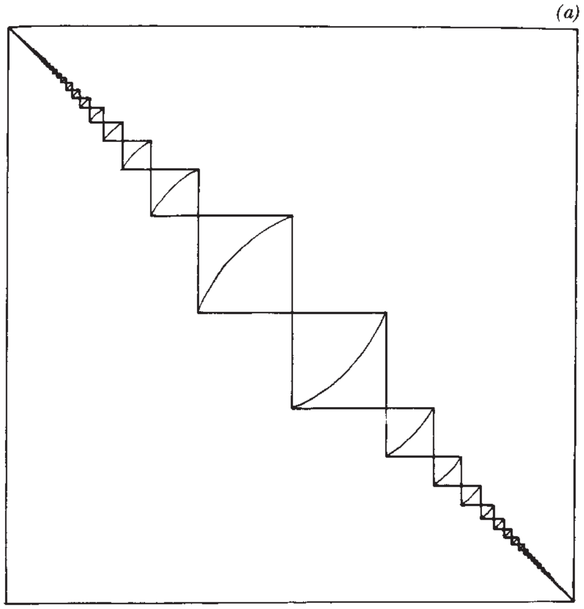
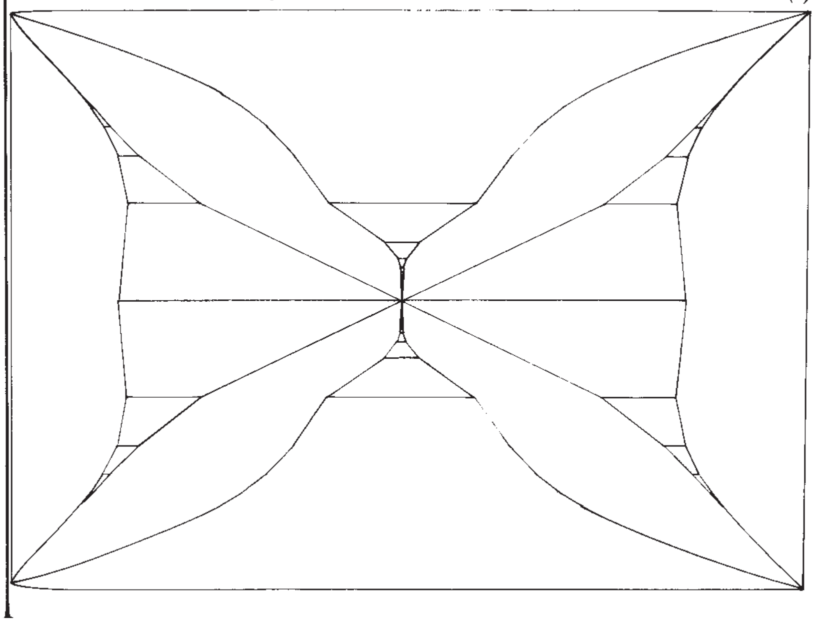


FIGURE 33(a) The skeleton from which INT can be constructed by recursive substitutions.
(b) The skeleton from which Gplot can be constructed by recursive substitutions.



Gplot shows that distribution. The horizontal axis represents energy, and the vertical axis represents the above-mentioned ratio of time periods, which we can call "α". At the bottom, α is zero, and at the top α is unity. When α is zero, there is no magnetic field. Each of the line segments making up Gplot is an "energy band"—that is, it represents allowed values of energy. The empty swaths traversing Gplot on all different size scales are therefore regions of forbidden energy. One of the most startling properties of Gplot is that when α is rational (say p/q in lowest terms), there are exactly q such bands (though when q is even, two of them "kiss" in the middle). And when α is irrational, the bands shrink to points, of which there are infinitely many, very sparsely distributed in a so-called "Cantor set"—another recursively defined entity which springs up in topology.

You might well wonder whether such an intricate structure would ever show up in an experiment. Frankly, I would be the most surprised person in the world if Gplot came out of any experiment. The physicality of Gplot lies in the fact that it points the way to the proper mathematical treatment of less idealized problems of this sort. In other words, Gplot is purely a contribution to theoretical physics, not a hint to experimentalists as to what to expect to see! An agnostic friend of mine once was so struck by Gplot's infinitely many infinities that he called it "a picture of God", which I don't think is blasphemous at all.

Recursion at the Lowest Level of Matter

We have seen recursion in the grammars of languages, we have seen recursive geometrical trees which grow upwards forever, and we have seen one way in which recursion enters the theory of solid state physics. Now we are going to see yet another way in which the whole world is built out of recursion. This has to do with the structure of elementary particles: electrons, protons, neutrons, and the tiny quanta of electromagnetic radiation called "photons". We are going to see that particles are—in a certain sense which can only be defined rigorously in relativistic quantum mechanics—nested inside each other in a way which can be described recursively, perhaps even by some sort of "grammar".

We begin with the observation that if particles didn't interact with each other, things would be incredibly simple. Physicists would like such a world because then they could calculate the behavior of all particles easily (if physicists in such a world existed, which is a doubtful proposition). Particles without interactions are called *bare particles*, and they are purely hypothetical creations; they don't exist.

Now when you "turn on" the interactions, then particles get tangled up together in the way that functions F and M are tangled together, or married people are tangled together. These real particles are said to be *renormalized*—an ugly but intriguing term. What happens is that no particle can even be defined without referring to all other particles, whose definitions in turn depend on the first particles, etc. Round and round, in a never-ending loop.

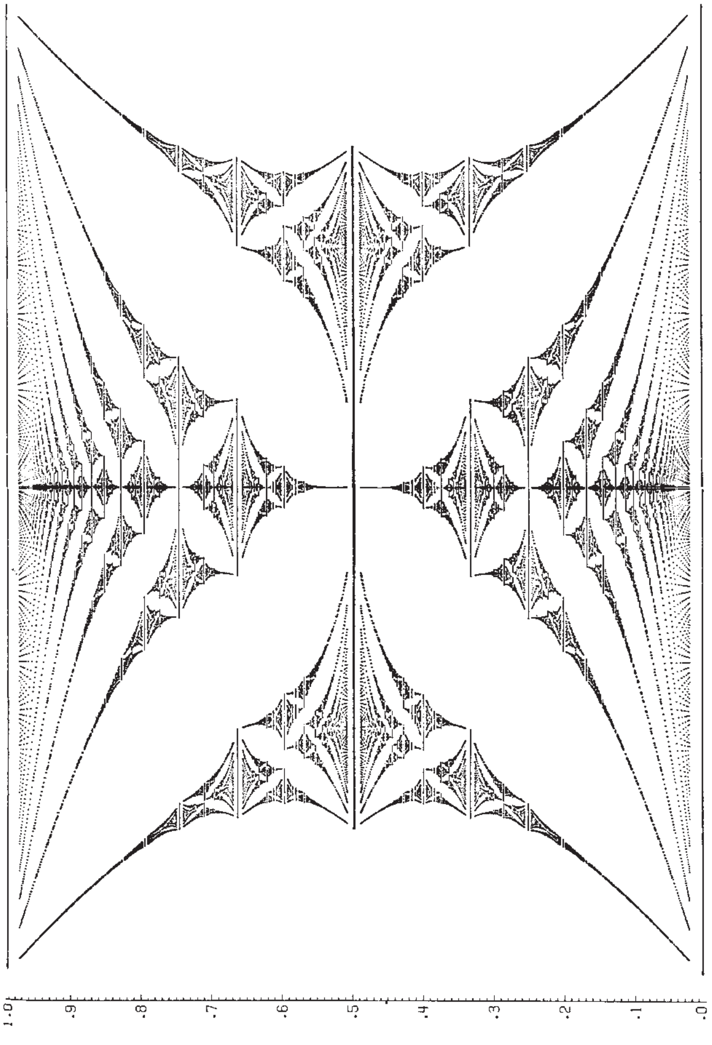
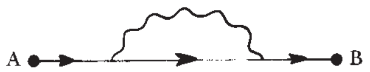


FIGURE 34. Gplot: a recursive graph showing energy bands for electrons in an idealized crystal in a magnetic field. α, representing magnetic field strength, runs vertically from 0 to 1. Energy runs horizontally. The horizontal line segments are bands of allowed electron energies.

Let us be a little more concrete, now. Let's limit ourselves to only two kinds of particles: *electrons* and *photons*. We'll also have to throw in the electron's antiparticle, the *positron*. (Photons are their own antiparticles.) Imagine first a dull world where a bare electron wishes to propagate from point A to point B, as Zeno did in my *Three-Part Invention*. A physicist would draw a picture like this:

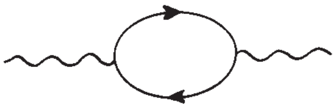
There is a mathematical expression which corresponds to this line and its endpoints, and it is easy to write down. With it, a physicist can understand the behavior of the bare electron in this trajectory.

Now let us "turn on" the electromagnetic interaction, whereby electrons and photons interact. Although there are no photons in the scene, there will nevertheless be profound consequences even for this simple trajectory. In particular, our electron now becomes capable of emitting and then reabsorbing *virtual photons*—photons which flicker in and out of existence before they can be seen. Let us show one such process:



Now as our electron propagates, it may emit and reabsorb one photon after another, or it may even nest them, as shown below:

The mathematical expressions corresponding to these diagrams—called "Feynman diagrams"—are easy to write down, but they are harder to calculate than that for the bare electron. But what really complicates matters is that a photon (real or virtual) can decay for a brief moment into an electron-positron pair. Then these two annihilate each other, and, as if by magic, the original photon reappears. This sort of process is shown below:



The electron has a right-pointing arrow, while the positron's arrow points leftwards.

As you might have anticipated, these virtual processes can be nested inside each other to arbitrary depth. This can give rise to some very complicated-looking drawings, such as the one in Figure 35. In that Feynman diagram, a single electron enters on the left at A, does some amazing acrobatics, and then a single electron emerges on the right at B. To an outsider who can't see the inner mess, it looks as if one electron has peacefully sailed from A to B. In the diagram, you can see how electron lines can get arbitrarily embellished, and so can the photon lines. This diagram would be ferociously hard to calculate.

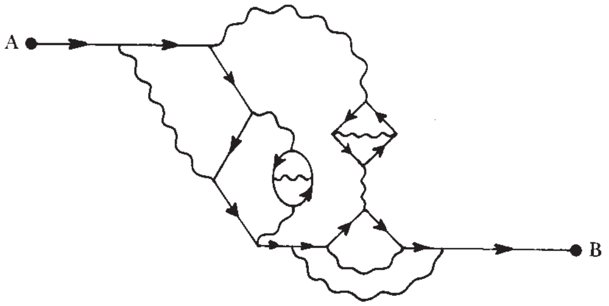


FIGURE 35. A Feynman diagram showing the propagation of a renormalized electron from A to B. In this diagram, time increases to the right. Therefore, in the segments where the electron's arrow points leftwards, it is moving "backwards in time". A more intuitive way to say this is that an antielectron (positron) is moving forwards in time. Photons are their own antiparticles; hence their lines have no need of arrows.

There is a sort of "grammar" to these diagrams, that only allows certain pictures to be realized in nature. For instance, the one below is impossible:

You might say it is not a "well-formed" Feynman diagram. The grammar is a result of basic laws of physics, such as conservation of energy, conservation of electric charge, and so on. And, like the grammars of human languages, this grammar has a recursive structure, in that it allows deep nestings of structures inside each other. It would be possible to draw up a set of recursive transition networks defining the "grammar" of the electromagnetic interaction.

When bare electrons and bare photons are allowed to interact in these arbitrarily tangled ways, the result is *renormalized* electrons and photons. Thus, to understand how a real, physical electron propagates from A to B, the physicist has to be able to take a sort of average of all the infinitely many different possible drawings which involve virtual particles. This is Zeno with a vengeance!

Thus the point is that a physical particle—a renormalized particle—involves (1) a bare particle and (2) a huge tangle of virtual particles, inextricably wound together in a recursive mess. Every real particle's existence therefore involves the existence of infinitely many other particles, contained in a virtual "cloud" which surrounds it as it propagates. And each of the virtual particles in the cloud, of course, also drags along its own virtual cloud, and so on ad infinitum.

Particle physicists have found that this complexity is too much to handle, and in order to understand the behavior of electrons and photons, they use approximations which neglect all but fairly simple Feynman diagrams. Fortunately, the more complex a diagram, the less important its contribution. There is no known way of summing up all of the infinitely many possible diagrams, to get an expression for the behavior of a fully renormalized, physical electron. But by considering roughly the simplest hundred diagrams for certain processes, physicists have been able to predict one value (the so-called g-factor of the muon) to nine decimal places—correctly!

Renormalization takes place not only among electrons and photons. Whenever any types of particle interact together, physicists use the ideas of renormalization to understand the phenomena. Thus protons and neutrons, neutrinos, pi-mesons, quarks—all the beasts in the subnuclear zoo—they all have bare and renormalized versions in physical theories. And from billions of these bubbles within bubbles are all the beasts and baubles of the world composed.

Copies and Sameness

Let us now consider Gplot once again. You will remember that in the Introduction, we spoke of different varieties of canons. Each type of canon exploited some manner of taking an original theme and copying it by an isomorphism, or information-preserving transformation. Sometimes the copies were upside down, sometimes backwards, sometimes shrunken or expanded . . . In Gplot we have all those types of transformation, and more. The mappings between the full Gplot and the "copies" of itself inside itself involve size changes, skewings, reflections, and more. And yet there remains a sort of skeletal identity, which the eye can pick up with a bit of effort, particularly after it has practiced with INT.

Escher took the idea of an object's parts being copies of the object itself and made it into a print: his woodcut *Fishes and Scales* (Fig. 36). Of course these fishes and scales are the same only when seen on a sufficiently abstract plane. Now everyone knows that a fish's scales aren't really small copies of the fish; and a fish's cells aren't small copies of the fish; however, a fish's DNA, sitting inside each and every one of the fish's cells, *is* a very convoluted "copy" of the entire fish—and so there is more than a grain of truth to the Escher picture.

FIGURE 36. Fish and Scales, *by M. C. Escher (woodcut, 1959).*

What is there that is the "same" about all butterflies? The mapping from one butterfly to another does not map cell onto cell; rather, it maps functional part onto functional part, and this may be partially on a macroscopic scale, partially on a microscopic scale. The exact proportions of parts are not preserved; just the functional relationships between parts. That is the type of isomorphism which links all butterflies in Escher's wood engraving *Butterflies* (Fig. 37) to each other. The same goes for the more abstract butterflies of Gplot, which are all linked to each other by mathematical mappings that carry functional part onto functional part, but totally ignore exact line proportions, angles, and so on.

Taking this exploration of sameness to a yet higher plane of abstraction, we might well ask, "What is there that is the 'same' about all Escher drawings?" It would be quite ludicrous to attempt to map them piece by piece onto each other. The amazing thing is that even a tiny section of an

FIGURE 37. Butterflies, *by M. C. Escher (wood-engraving, 1950).*

Escher drawing or a Bach piece gives it away. Just as a fish's DNA is contained inside every tiny bit of the fish, so a creator's "signature" is contained inside every tiny section of his creations. We don't know what to call it but "style"—a vague and elusive word.

We keep on running up against "sameness-in-differentness", and the question

When are two things the same?

It will recur over and over again in this book. We shall come at it from all sorts of skew angles, and in the end, we shall see how deeply this simple question is connected with the nature of intelligence.

That this issue arose in the Chapter on recursion is no accident, for recursion is a domain where "sameness-in-differentness" plays a central role. Recursion is based on the "same" thing happening on several different levels at once. But the events on different levels *aren't* exactly the same—rather, we find some invariant feature in them, despite many ways in which they differ. For example, in the *Little Harmonic Labyrinth*, all the stories on different levels are quite unrelated—their "sameness" resides in only two facts: (1) they are stories, and (2) they involve the Tortoise and Achilles. Other than that, they are radically different from each other.

Programming and Recursion: Modularity, Loops, Procedures

One of the essential skills in computer programming is to perceive when two processes are the same in this extended sense, for that leads to *modularization*—the breaking-up of a task into natural subtasks. For instance, one might want a sequence of many similar operations to be carried out one after another. Instead of writing them all out, one can write a *loop,* which tells the computer to perform a fixed set of operations and then loop back and perform them again, over and over, until some condition is satisfied. Now the *body* of the loop—the fixed set of instructions to be repeated—need not actually be completely fixed. It may vary in some predictable way.

An example is the most simple-minded test for the primality of a natural number N, in which you begin by trying to divide N by 2, then by 3, 4, 5, etc. until $N-1$. If N has survived all these tests without being divisible, it's prime. Notice that each step in the loop is similar to, but not the same as, each other step. Notice also that the number of steps varies with N—hence a loop of fixed length could never work as a general test for primality. There are two criteria for "aborting" the loop: (1) if some number divides N exactly, quit with answer "NO"; (2) if $N-1$ is reached as a test divisor and N survives, quit with answer "YES".

The general idea of loops, then, is this: perform some series of related steps over and over, and abort the process when specific conditions are met. Now sometimes, the maximum number of steps in a loop will be known in advance; other times, you just begin, and wait until it is aborted. The second type of loop—which I call a *free* loop—is dangerous, because the criterion for abortion may never occur, leaving the computer in a so-called "infinite loop". This distinction between *bounded loops* and *free loops* is one of the most important concepts in all of computer science, and we shall devote an entire Chapter to it: "BlooP and FlooP and GlooP".

Now loops may be nested inside each other. For instance, suppose that we wish to test all the numbers between 1 and 5000 for primality. We can write a second loop which uses the above-described test over and over, starting with $N = 1$ and finishing with $N = 5000$. So our program will have a "loop-the-loop" structure. Such program structures are typical—in fact they are deemed to be good programming style. This kind of nested loop also occurs in assembly instructions for commonplace items, and in such activities as knitting or crocheting—in which very small loops are repeated several times in larger loops, which in turn are carried out repeatedly . . . While the result of a low-level loop might be no more than couple of stitches, the result of a high-level loop might be a substantial portion of a piece of clothing.

In music, too, nested loops often occur—as, for instance, when a scale (a small loop) is played several times in a row, perhaps displaced in pitch each new time. For example, the last movements of both the Prokofiev fifth piano concerto and the Rachmaninoff second symphony contain extended passages in which fast, medium, and slow scale-loops are played simultaneously by different groups of instruments, to great effect. The Prokofiev-scales go up; the Rachmaninoff-scales, down. Take your pick.

A more general notion than loop is that of *subroutine,* or *procedure,* which we have already discussed somewhat. The basic idea here is that a group of operations are lumped together and considered a single unit with a name—such as the procedure **ORNATE NOUN**. As we saw in RTN's, procedures can call each other by name, and thereby express very concisely sequences of operations which are to be carried out. This is the essence of modularity in programming. Modularity exists, of course, in hi-fi systems, furniture, living cells, human society—wherever there is hierarchical organization.

More often than not, one wants a procedure which will act variably, according to context. Such a procedure can either be given a way of peering out at what is stored in memory and selecting its actions accordingly, or it can be explicitly fed a list of *parameters* which guide its choice of what actions to take. Sometimes both of these methods are used. In RTN-terminology, choosing the sequence of actions to carry out amounts to *choosing which pathway to follow.* An RTN which has been souped up with parameters and conditions that control the choice of pathways inside it is called an *Augmented Transition Network* (ATN). A place where you might prefer ATN's to RTN's is in producing sensible—as distinguished from nonsensical—English sentences out of raw words, according to a grammar represented in a set of ATN's. The parameters and conditions would allow you to insert various semantic constraints, so that random juxtapositions like "a thankless brunch" would be prohibited. More on this in Chapter XVIII, however.

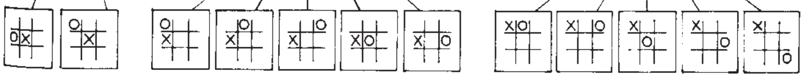
Recursion in Chess Programs

A classic example of a recursive procedure with parameters is one for choosing the "best" move in chess. The best move would seem to be the one which leaves your opponent in the toughest situation. Therefore, a test for goodness of a move is simply this: pretend you've made the move, and now evaluate the board from the point of view of your opponent. But how does your opponent evaluate the position? Well, he looks for *his* best move. That is, he mentally runs through all possible moves and evaluates them from what he thinks is *your* point of view, hoping they will look bad to you. But notice that we have now defined "best move" recursively, simply using the maxim that what is best for one side is worst for the other. The recursive procedure which looks for the best move operates by trying a move, and then *calling on itself in the role of opponent!* As such, it tries another move, and calls on itself in the role of its opponent's opponent—that is, itself.

This recursion can go several levels deep—but it's got to bottom out somewhere! How do you evaluate a board position *without* looking ahead? There are a number of useful criteria for this purpose, such as simply the number of pieces on each side, the number and type of pieces under attack, the control of the center, and so on. By using this kind of evaluation at the bottom, the recursive move-generator can pop back upwards and give an evaluation at the top level of each different move. One of the parameters in the self-calling, then, must tell how many moves to look ahead. The outermost call on the procedure will use some externally set value for this parameter. Thereafter, each time the procedure recursively calls itself, it must decrease this look-ahead parameter by 1. That way, when the parameter reaches zero, the procedure will follow the alternate pathway—the non-recursive evaluation.

In this kind of game-playing program, each move investigated causes the generation of a so-called "look-ahead tree", with the move itself as trunk, responses as main branches, counter-responses as subsidiary branches, and so on. In Figure 38 I have shown a simple look-ahead tree, depicting the start of a tic-tac-toe game. There is an art to figuring out how to avoid exploring every branch of a look-ahead tree out to its tip. In chess trees, people—not computers—seem to excel at this art; it is known that top-level players look ahead relatively little, compared to most chess programs—yet the people are far better! In the early days of computer chess, people used to estimate that it would be ten years until a computer (or

FIGURE 38. The branching tree of moves and countermoves at the start of a game of tic-tac-toe.



program) was world champion. But after ten years had passed, it seemed that the day a computer would become world champion was still more than ten years away . . . This is just one more piece of evidence for the rather recursive

> *Hofstadter's Law:* It always takes longer than you expect, even when you take into account Hofstadter's Law.

Recursion and Unpredictability

Now what is the connection between the recursive processes of this Chapter, and the recursive sets of the preceding Chapter? The answer involves the notion of a *recursively enumerable set.* For a set to be r.e. means that it can be generated from a set of starting points (axioms), by the repeated application of rules of inference. Thus, the set grows and grows, each new element being compounded somehow out of previous elements, in a sort of "mathematical snowball". But this is the essence of recursion—something being defined in terms of simpler versions of itself, instead of explicitly. The Fibonacci numbers and the Lucas numbers are perfect examples of r.e. sets—snowballing from two elements by a recursive rule into infinite sets. It is just a matter of convention to call an r.e. set whose complement is also r.e. "recursive".

Recursive enumeration is a process in which new things emerge from old things by fixed rules. There seem to be many surprises in such processes—for example the unpredictability of the Q-sequence. It might seem that recursively defined sequences of that type possess some sort of inherently increasing complexity of behavior, so that the further out you go, the less predictable they get. This kind of thought carried a little further suggests that suitably complicated recursive systems might be strong enough to break out of any predetermined patterns. And isn't this one of the defining properties of intelligence? Instead of just considering programs composed of procedures which can recursively *call* themselves, why not get really sophisticated, and invent programs which can *modify* themselves—programs which can act on programs, extending them, improving them, generalizing them, fixing them, and so on? This kind of "tangled recursion" probably lies at the heart of intelligence.