Lecture 24: Gödel’s Proof

David Evans
http://www.cs.virginia.edu/evans

PS6 Classes

- sim-object
- physical-object
- place
- mobile-object
- thing
- person
- student
- police-officer

Are there class hierarchies like this in the “real world” or just in fictional worlds like Charlottsville?

Microsoft Foundation Classes

- CButton
- CWnd
- CObject

CBButton inherits from CWnd inherits from CObject
“A button is a kind of window is a kind of object”

Java 3D Class Hierarchy Diagram

-not at all uncommon to have class hierarchies like this!

Quiz?


Story So Far

- Much of the course so far:
  - Getting comfortable with recursive definitions
  - Learning to write a program to do (almost) anything (PS1-4)
  - Learning more elegant ways of programming (PS5-6)

- This Week:
  - Getting **un**-comfortable with recursive definitions
  - Understanding why there are some things no program can do!
Lecture 24: Gödel’s Proof

Computer Science/Mathematics

• Computer Science (Imperative Knowledge)
  – Are there (well-defined) problems that cannot be solved by any procedure?

• Mathematics (Declarative Knowledge)
  – Are there true conjectures that cannot be shown using any proof?

Today

Wednesday

Mechanical Reasoning

Aristotle (~350BC): Organon
Codify logical deduction with rules of inference (syllogisms)

\[
\frac{\text{Every } A \text{ is a } P}{\text{X is an } A} \quad \frac{\text{X is a } P}{\text{Conclusion}}
\]

Every human is mortal.

\[
\text{Gödel is human.}
\]

Gödel is mortal.

More Mechanical Reasoning

• Euclid (~300BC): Elements
  – We can reduce geometry to a few axioms and derive the rest by following rules

• Newton (1687): Philosophiae Naturalis Principia Mathematica
  – We can reduce the motion of objects (including planets) to following axioms (laws) mechanically

Mechanical Reasoning

• Late 1800s – many mathematicians working on codifying “laws of reasoning”
  – George Boole, Laws of Thought
  – Augustus De Morgan

• Whitehead and Russell, 1911-1913
  – Principia Mathematica
  – Attempted to formalize all mathematical knowledge about numbers and sets

Perfect Axiomatic System

Derives all true statements, and no false statements starting from a finite number of axioms and following mechanical inference rules.
**Incomplete Axiomatic System**

Derives some, but not all true statements, and no false statements starting from a finite number of axioms and following mechanical inference rules.

**Inconsistent Axiomatic System**

Derives all true statements, and some false statements starting from a finite number of axioms and following mechanical inference rules.

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**Principia Mathematica**

- Whitehead and Russell (1910–1913)
  - Three Volumes, 2000 pages
- Attempted to axiomatize mathematical reasoning
  - Define mathematical entities (like numbers) using logic
  - Derive mathematical "truths" by following mechanical rules of inference
  - Claimed to be complete and consistent
    - All true theorems could be derived
    - No falsehoods could be derived

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**Russell’s Paradox**

- Some sets are not members of themselves
  - set of all Jeffersonians
- Some sets are members of themselves
  - set of all things that are non-Jeffersonian
- $S$ = the set of all sets that are not members of themselves
  - Is $S$ a member of itself?

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**Ban Self-Reference?**

- *Principia Mathematica* attempted to resolve this paragraph by banning self-reference
  - Every set has a type
    - The lowest type of set can contain only "objects", not "sets"
    - The next type of set can contain objects and sets of objects, but not sets of sets
Russell’s Resolution?

Set ::= Set

Set₀ ::= \{ x | x \text{ is an Object} \}

Setₙ ::= \{ x | x \text{ is an Object or a } Set_{n-1} \}

S: Setₙ

Is S a member of itself?

No, it is a Setₙ so, it can’t be a member of a Setₙ.

Epimenides Paradox

Epidenides (a Cretan):

“All Cretans are liars.”

Equivalently:

“This statement is false.”

Russell’s types can help with the set paradox, but not with these.

Gödel’s Solution

All consistent axiomatic formulations of number theory include undecidable propositions.

(\text{GEB, p. 17})

undecidable – cannot be proven either true or false inside the system.

Kurt Gödel

• Born 1906 in Brno (now Czech Republic, then Austria-Hungary)
• 1931: publishes Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme (On Formally Undecidable Propositions of Principia Mathematica and Related Systems)
• 1939: flees Vienna
• Institute for Advanced Study, Princeton
• Died in 1978 – convinced everything was poisoned and refused to eat

Gödel’s Theorem

In the Principia Mathematica system, there are statements that cannot be proven either true or false.
Gödel’s Theorem

In any interesting rigid system, there are statements that cannot be proven either true or false.

Gödel’s Theorem

All logical systems of any complexity are incomplete: there are statements that are true that cannot be proven within the system.

Proof – General Idea

• Theorem: In the Principia Mathematica system, there are statements that cannot be proven either true or false.
• Proof: Find such a statement

Gödel’s Statement

$G$: This statement does not have any proof in the system of Principia Mathematica.

$G$ is unprovable, but true!

Gödel’s Proof Idea

$G$: This statement does not have any proof in the system of PM.

If $G$ is provable, PM would be inconsistent.
If $G$ is unprovable, PM would be incomplete.

Thus, PM cannot be complete and consistent!

Charge

• Wednesday:
  – Finish the proof: show we can express $G$
  – What is the equivalent to the Gödel sentence for computation?
• Friday:
  – How to prove a problem has no solving procedure
• Next Monday:
  – History of Object-Oriented Programming