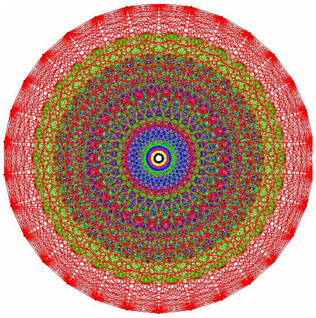
Lecture 26: Proving Uncomputability



Visualization of E8

CS150: Computer Science
University of Virginia
Computer Science

David Evans
http://www.cs.virginia.edu/evans

The Halting Problem

Input: a specification of a procedure *P*

Output: If evaluating an application of *P* halts, output true. Otherwise, output false.

halts? Examples

```
> (halts? '(lambda () (+ 3 3)))
#t
> (halts? '(lambda () (define (f) (f)) (f)))
#f
> (halts? `(lambda ()
            (define (fibo n)
              (if (or (= n 1) (- n 2))) 1
                 (+ (fibo (- n 1)) (fibo (- n 2))))))
            (fibo 100))
#t
```


Halting Examples

```
> (halts? `(lambda ()
             (define (sum-of-two-primes? n)
                ;;; try all possibilities... )
             (define (test-goldbach n)
                 (if (not (sum-of-two-primes? n))
                     #f ; Goldbach Conjecture wrong
                     (test-goldbach (+ n 2))))
              (test-goldbach 2))
?
```

Goldbach Conjecture (see GEB, p. 394):
Every even integer can be written as the sum of two primes.

Can we define halts? ?

- We could try for a really long time, get something to work for simple examples, but could we solve the problem – make it work for all possible inputs?

Informal Proof

```
(define (paradox)
  (if (halts? 'paradox)
      (loop-forever)
      #t))
```

If paradox halts, the if test is true and it evaluates to (loop-forever) - it doesn't halt!

If paradox doesn't halt, the if test if false, and it evaluates to #t. It halts!

Proof by Contradiction

1. Show *X* is nonsensical.
2. Show that if you have *A* you can make *X*.
3. Therefore, *A* must not exist.

X = paradox
A = halts? algorithm

How convincing is our Halting Problem proof?

```
(define (paradox)
  (if (halts? 'paradox)
      (loop-forever)
      #t))
```

If contradict-halts halts, the if test is true and it evaluates to (loop-forever) - it doesn't halt!
If contradict-halts doesn't halt, the if test if false, and it evaluates to #t. It halts!

This "proof" assumes Scheme exists and is consistent! Scheme is too complex to believe this...we need a simpler model of computation (in two weeks).

Evaluates-to-3 Problem

Input: A procedure specification *P*
Output: **true** if evaluating (*P*) would result in 3; **false** otherwise.

Is "Evaluates to 3" computable?

Proof by Contradiction

1. Show *X* is nonsensical.
2. Show that if you have *A* you can make *X*.
3. Therefore, *A* must not exist.

X = halts? algorithm
A = evaluates-to-3? algorithm

Undecidability Proof

Suppose we could define evaluates-to-3? that decides it. Then we could define halts?:

```
(define (halts? P)
  (evaluates-to-3?
    '(lambda () (begin (P) 3))))
```

if #t: it evaluates to 3, so we know (P) must halt.

if #f: the only way it could not evaluate to 3, is if (P) doesn't halt. (Note: assumes (P) cannot produce an error.)

Hello-World Problem

Input: An expression specification *E*
Output: **true** if evaluating *E* would print out "Hello World!"; **false** otherwise.

Is the *Hello-World Problem* computable?

Uncomputability Proof

Suppose we could define prints-hello-world? that solves it. Then we could define halts?:

```
(define (halts? P)
   (prints-hello-world?
      '(begin ((remove-prints P))
                  (print "Hello World!"))))
```


Proof by Contradiction

1. Show *X* is nonsensical.
2. Show that if you have *A* you can make *X*.
3. Therefore, *A* must not exist.

X = halts? algorithm
A = prints-hello-world? algorithm

Charge

- Next week:
  - Monday: computability of virus detection, AliG problem; history of Object-Oriented programming
  - Wednesday, Friday: implementing interpreters
- After next week:
  - Scheme is very complicated (requires more than 1 page to define)
  - To have a convincing proof, we need a simpler programming model in which we can write paradox: Turing's model