Liuyi Zhang
Challenge Problem Solution

**Theorem:** $S_{\text{Even}}$ as defined below, generates all even strings that are not in $L_{ww}$.

1. $S_{\text{Even}} \rightarrow XY \mid YX$
2. $X \rightarrow ZXZ \mid 0$
3. $Y \rightarrow ZYZ \mid 1$
4. $Z \rightarrow 0 \mid 1$

**Lemma 1.** $S_{\text{Even}}$ can generate all the strings length equal to $2n$ ($n > 0$, $n$ is integer) in the following two forms:

$$Z^j 0 Z^j 1 Z^k \text{ and } Z^j 1 Z^j 0 Z^k$$

where $j, k$ are integers, $j \geq 0$, $k \geq 0$, and $2j+2k+2 = 2(j+k+1) = 2n$. This follows from the grammar rules. The first form corresponds to derivations of the form:

$$S_{\text{Even}} \rightarrow XY \rightarrow ZXZY \rightarrow \ldots \rightarrow Z^j X Z^j Y \rightarrow Z^j 0 Z^j Y \rightarrow \ldots \rightarrow Z^j 0 Z^j 1 Z^j \rightarrow Z^j 0 Z^j 1 Z^k$$

($j$ uses of rule 2a) (by rule 2b) ($k$ uses of rule 3a) (using rule 3b)

The second form corresponds similarly to derivations that start with $S_{\text{Even}} \rightarrow YX$.

Since each $Z$ produces either 0 or 1, we can equivalently write the two forms above by replacing $Z^j Z^k$ with $Z^j Z^k$s: $Z^j 0 Z^j 1 Z^k$ and $Z^j 1 Z^j 0 Z^k$.

**Lemma 2.** We know the grammar could only generate strings in the following format $Z^j 0 Z^j 1 Z^k$ or $Z^j 1 Z^j 0 Z^k$.

**Proof.** With only $X$ productions $X \rightarrow ZXZ \mid 0$ we can only generate strings in the following format by repeating using the grammar. As there is no other production for $X$, we can conclude that $X = Z^j Z^k$

Similarly, with only $Y$ productions $Y \rightarrow ZYZ \mid 1$ we can only generate strings in the following format $Y = Z^j 1 Z^k$ by repeating using the grammar. As there is no other rule for $Y$, we can conclude that $Y = Z^j 1 Z^k$

Combining both parts, with grammar $S_{\text{Even}} \rightarrow XY \mid YX$ we know that all strings produced by $S_{\text{Even}}$ could only be the following format $Z^j 0 Z^j 1 Z^k$ or $Z^j 1 Z^j 0 Z^k$.

**Proof:** To prove that $S_{\text{Even}}$ generates $L_{ww}$, the language of all even-length strings that are not composed of two matching halves, we observe that for any string $s \in L_{ww}$ there must be some position $i$ in the string where the value of $s[i]$ is different from the value of $s[|s|/2 + i]$. We show that $S_{\text{Even}}$
can generate all such strings, and generates no strings without a mismatch at some position. There are only two symbols in the alphabet, so these two cases cover all possibilities: \( s[i] = 0 \) and \( s[i] = 1 \) where \( 0 \leq i < |s|/2 \).

**Case 1.** \( s[i] = 0 \).

First, we show the grammar produces all of the even-length strings in \( L_{ww} \) where there is some \( i, 0 \leq i < |s|/2 \), such that \( s[i] = 0 \) and \( s[|s|/2 + i] = 1 \). All even length strings \( t \) of length \( 2n \) can be divided into two equal length strings of length \( n: t = t_1t_2 \). Since we are covering the case where \( s[i] = 0 \), and \( s[|s|/2 + i] = 1 \), all strings have the form \( t = Z0Z^i1Z^k \) where \( n = i + k + 1 \). From the lemma, \( S_{\text{Even}} \) generates all such strings by substituting \( i \) and \( j \).

Now, we show that the grammar does not produce any strings in \( L_{ww} \). We prove by contradiction: if \( t \) is in \( L_{ww} \), then \( t_1 \) must equal to \( t_2 \). We know the grammar could only generate strings in the following format \( Z0Z^i1Z^k \) or \( Z1Z^i0Z^k \) (from Lemma 2). If we separate the string into two equal length strings \( t = t_1t_2 \), we can only separate it as \( t_1 = Z0Z^i \) and \( t_2 = Z1Z^k \), then, in \( t_1 \) there must be \( j \) symbols in front of the 0; same in \( t_2 \), there are \( j \) symbols in font of the 1. As there will be different alphabet appear on the same position, therefore \( t_1 \) always NOT equal to \( t_2 \). Thus, we have a contradiction.

**Case 2.** \( s[i] = 1 \). This follows identically to case 1, except using the second form corresponding to the \( S_{\text{Even}} \rightarrow YX \) rule.