CS447/647: Image Synthesis Lecture #6
University of Virginia Tuesday, 11 February 2003

Introduction

Lecture #6: Tuesday, 4 February 2003
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1 Measuring sets of rays

Definition: A Beam of rays is the set of rays intersecting two surface elements (see Figure **??**).

The beam characterizes a bundle of rays. What we would like to be able to find out is how many there are. For this we will use the notion of throughput, which we will define as dT. Throughput will serve as a measure of the number of rays entering or leaving a surface.

$$dT = \frac{dA_1 \cdot dA_2}{r^2} = dA_1 \cdot d\omega_1 = dA_2 \cdot d\omega_2$$

We can also do this with tilted surfaces, as in Figure **??**. In this case, our definition of throughput is changed to

$$dT = \frac{dA_1 \cdot dA_2}{r^2} \cdot cod\theta_1 \cdot cos\theta_2$$

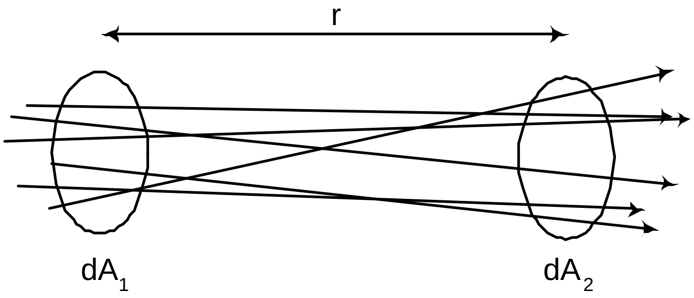

Figure 1: A beam of rays through parallel differential areas.

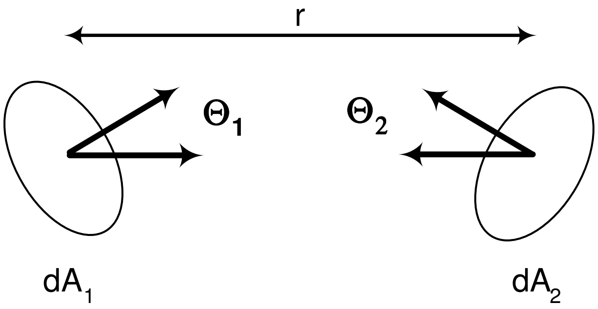

Figure 2: A beam of rays through arbitrary differential areas.

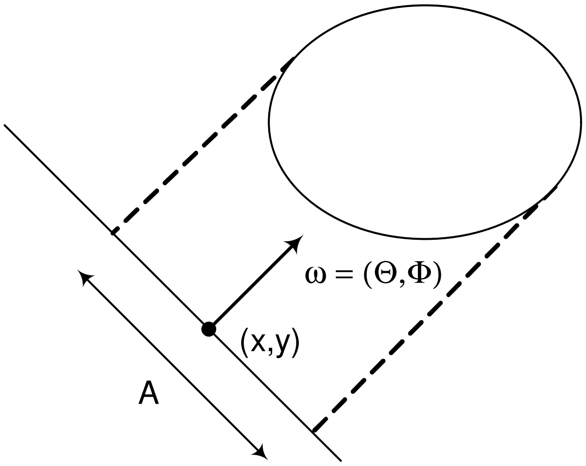

Figure 3: Parameterization of rays with respect to a plane and a direction normal to the plane.

2 Parameterizing Rays

What we will really care about is the projected area of an object onto a plane. First we will define one type of paramterization for a ray that will be useful. We want to define a ray by a direction $\omega = (\theta, \phi)$, and a plane normal to that direction (see Figure **??**). Then we can choose a point (x, y) on that plane, and we now have a new definition of a ray, (x, y, θ, ϕ).

Now we can find the total number of rays intersecting an object. All we need to do is integrate over our measure:

$$\int_{S^2} \int_A d\omega \cdot dA$$

$$= \int_{S^2} d\omega \cdot \int_A dA = 4\pi \cdot \overline{\tilde{A}}$$

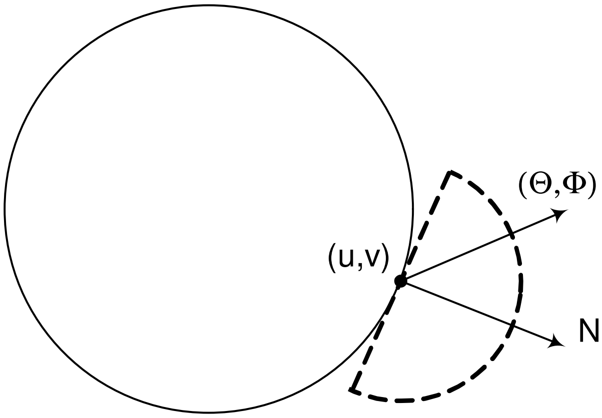

Figure 4: Parameterization of rays with respect to an object.

Here $\overline{\tilde{A}}$ is the average projected area, and as stated by Crofton's Theorem, $\overline{\tilde{A}} = \frac{S}{4}$, where S is the true surface area. This is not hard to prove. For exdample, take a sphere of radius r, throughout $T = 4\pi \cdot (\frac{4\pi r^2}{4}) = 4\pi^2 r^2 =$ the total number of rays that hit the sphere, which is 4π multiplied by the area of the sphere, πr^2 .

3 Projected Solid Angle

For projected solid angle, we will be parameterizing the ray in a different way. We will define a ray as $r = (u, v, \theta, \phi)$ as in Figure **??** . This only parameterizes rays that hit the object, there are many rays which can't be represented this way (all rays that miss the object, for instance).

Now we want to find the number of rays that hit this object. This is given as $\int_{m^2} dA \int_{H^2(N)} cos\theta d\omega$, where m^2 is any mainfold and $H^2(N)$ is the hemisphere centered at the normal. Figure **??** shows the geometric interpretation of this. The second part of the integral given above, $\int_{H^2(N)} cos\theta d\omega$ is just the integral over the total projected area of the hemisphere, which is just equal to π. The first part, $\int_{m^2} dA$, is just the surface area for the object. Thus our integral evaluats to $S\pi$ (for a unit hemisphere). As a quick example, think of a sphere. The surface are for a sphere is $4\pi r^2$, so the integral evaluates to $4\pi^2 r^2$, just as we determined earlier.

Let's also consider the throughput from a differential source to a finite object, and let's call this a differential-finite beam. In this case we have a small differential area dA_1 and a finite object A_2 as in Figure **??**. We want to calculate the throughput from dA_1 to A_2. This is really just $T = [\int_{A_2} \frac{cos\theta_1 \cdot cos\theta_2}{r^2} \cdot dA_2]dA$

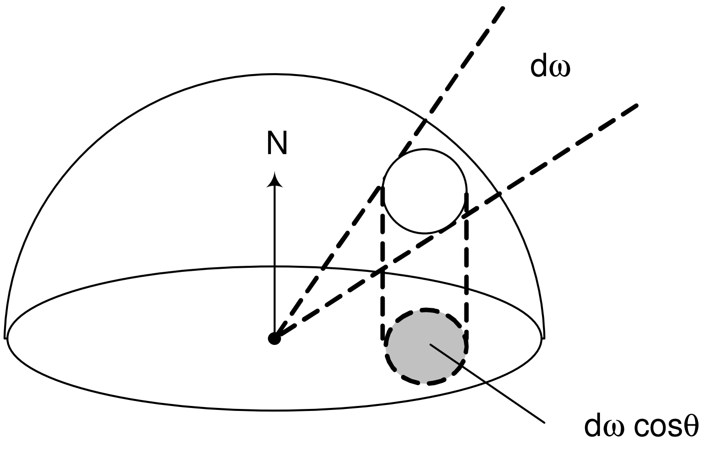

Figure 5: Projected solid angle (shaded area).

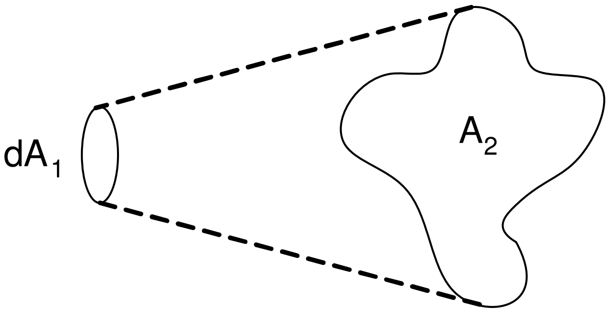

Figure 6: A Differential-finite beam.

4 Form Factors

We will also want to be able to determine the probability that any ray will intersect a given object. In Figure**??**, dA is the point of the ray origin, and A' is the object we will be concerned with. So we want to find $Pr(A'|dA) = \frac{T(dA,A')dA}{T\cdot dA}$. This is just throughput, we want to know the total number of objects in the beam between dA and A' divided by the total number of rays that hit dA. Here $T(dA,A')$ is just part of the equation we defined above for a differential-finite beam, $\int_{A'} \frac{cos\theta_1 \cdot cos\theta_2}{r^2} \cdot dA'$. We know $T \cdot dA$ is π times the surface area. We also have to be careful because we only want the rays leaving the top hemisphere, so the bottom of our equation is $\pi \cdot dA$. Our equation reduces to $\frac{1}{\pi} \int_{A'} \frac{cos\theta_1 \cdot cos\theta_2}{r^2} \cdot dA'$. This is our form factor. Since this is a pain to write, we usually define $G(x,x') = \frac{cos\theta_1 \cdot \cos\theta_2}{\pi|x-x'|^2}$ and the euqation becomes $\int_A' G \cdot dA'$.

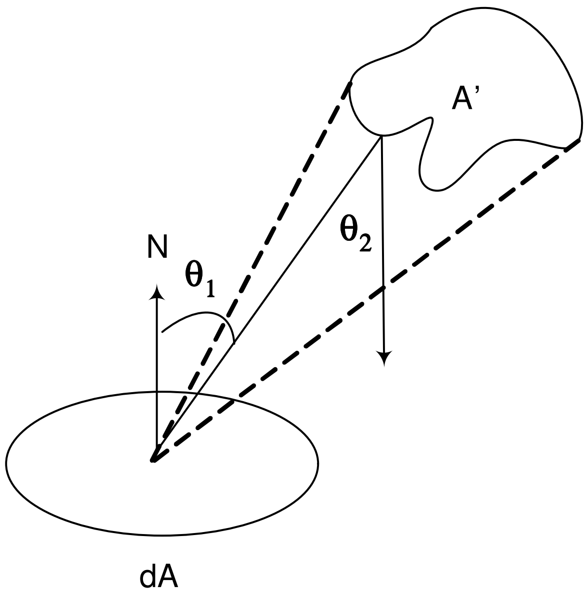

Figure 7: form Factors: We want to determine the probability that any ray leaving dA will intersect a'

5 Conservation of Throughput

Like power, throughput is a measure that is conserved. The number of rays never changes, rays don't dissappear and rays don't just appear. In fact, what is actually conserved is $\eta^2 \cdot T$ where η is the index of refraction at a point in space. Changing indices of refraction will change the total number of rays. At a refractive interface, the radiance will also change as measured along a ray.

This also means that radiance is conserved as well, since radiance is differential power divided by throughput:

$$L(r) = \lim_{\Delta T \to \infty} \frac{\Delta \Phi}{\Delta T}$$

6 Surface vs. Field Radiance

Surface radiance is intensity per projected areas per solid angle, given by $L(x, \omega) = \frac{d\Phi(x,\omega)}{d\omega dA cos\theta}$. Field radiance is power per solid angle per unit area perpendicular to the direction. It is really the same as surface radiance, just not at a surface.