Fourier Analysis

Lecture #08: Tuesday, 11 February 2003
Lecturer: Greg Humphreys
Scribe: John Tran
Reviewer: Joe Mama

This lecture covers the basics of Fourier Analysis.

1 Background

Images are made up of dots. This is true no matter what kind of image you are dealing with, be it paper, a monitor, photo film, a painting, etc. Therefore, we want as much spatial resolution as possible. This is however limited to our media, be it the screen's resolution or film granularity.

Because images are made up of dots, it is impossible to display the correct image. The image is merely a discrete sampling of a continuous function. This introduces errors known as *aliasing*. There errors arise because we are throwing away information. The trick is to be very careful with what you throw away.

To make matters worse, the human visual system fills in the data that we discard, so we need to be careful about how the viewer will interpret what is shown to them. We want the interpreted signal to match the original signal as closely as possible. The tool we use for analyzing the quality of this match is *Fourier Analysis*.

2 The Fourier Transform

A Fourier Transform respresents a signal in frequency space. This representation helps us understand what happens when a signal is turned into a bunch of samples. The Fourier Transform decomposes a signal into a weighted sum of sines and cosines. The Inverse Fourier Transform converts a signal in frequency space back into the time domain.

2.1 Basis Functions

Functions can be represented as a weighted combination of simpler *basis functions*. A simple example is points in Euclidean space, where our reference frame is the origin and three perpendicular unit vectors, x̂, ŷ, and ẑ. To represent any vector V, we specify three coordinates: V_x, V_y, and V_z. These are *weights*:

$$V = V_x\hat{x} + V_y\hat{y} + V_z\hat{z} = (V \bullet \hat{x})\hat{x} + (V \bullet \hat{y})\hat{y} + (V \bullet \hat{z})\hat{z} \tag{1}$$

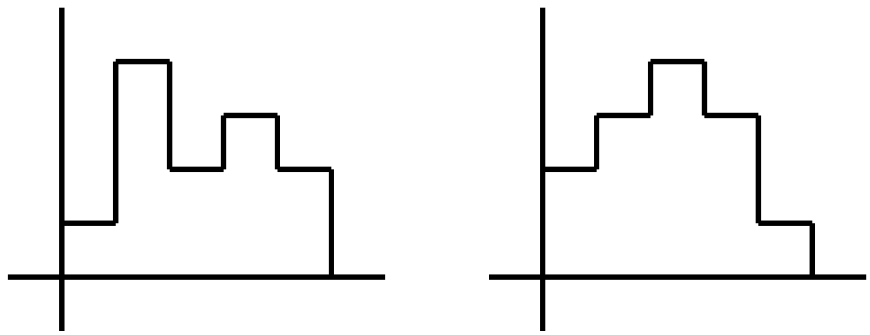

Figure 1: Example 5-unit bar chart functions (1,4,2,3,2) and (2,3,4,3,1)

2.2 Projecting Functions

Given any 1D function $y = f(x)$, basis functions $\Phi_i(x)$, and scalar weights c_i, we can write:

$$f(x) = c_1\Phi_1(x) + c_2\Phi_2(x) + \ldots + c_n\Phi_n(x) \tag{2}$$

For example, consider the following 5-unit bar chart functions in figure **??**.
The basis functions are:

$$\Phi_i = \begin{cases} 1 & i-1 \le t < i \\ 0 & otherwise \end{cases} \tag{3}$$

where $i \in \{1, 2, 3, 4, 5\}$.
Two questions that may arise about bases are:

1. How many do we need?
2. Is our set complete?

As an informal answer, it is good for basis vectors to be orthogonal.

2.3 Orthogonality

What does it mean for a pair of functions to be orthogonal?
Over some interval, $\Gamma = [t_1, t_2]$,

$$\int_{t_1}^{t_2} \Phi_i(t)\Phi_j(t)dt = \langle\Phi_i|\Phi_j\rangle_\Gamma \begin{cases} = 0 & i \neq j \\ \neq 0 & otherwise \end{cases} \tag{4}$$

(the bracket notation can be thought of as a generalized dot product)
So the projection of any function in the set onto any other function in the set is zero. Orthogonal sets are *normalized* if

$$\langle \Phi_i | \Phi_i \rangle_\Gamma = 1 \tag{5}$$

$$\langle \mu_i \Phi_i | \mu_i \Phi_i \rangle_\Gamma = 1 \tag{6}$$

$$\mu_i^2 \langle \Phi_i | \Phi_i \rangle_\Gamma = 1 \tag{7}$$

$$\mu_i = \frac{1}{\sqrt{\langle \Phi_i | \Phi_i \rangle_\Gamma}} \tag{8}$$

So how do we project a function f onto an orthogonal basis $\{\Phi_i\}$? Come up with a set of coefficients $\{c_i\}$ such that:

$$f(t) \approx \sum_{i=1}^{n} c_i \Phi_i(t) \tag{9}$$

Minimize the approximation error over some interval t_1 to t_2. This can be done by calculating the mean squared error MSE.

$$MSE = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(f(t) - \sum_{i=1}^{n} c_i \Phi_i(t) \right)^2 dt \tag{10}$$

$$MSE = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(f(t) - c_1 \Phi_1(t) - c_2 \Phi_2(t) - \ldots - c_n \Phi_n(t) \right)^2 dt \tag{11}$$

$$MSE = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (f_2(t) + c_1^2 \Phi_1^2(t) + c_2^2 \Phi_2^2(t) + \ldots + c_n^2 \Phi_n^2(t)$$

$$-2c_1 f(t)\Phi_1(t) - 2c_2 f(t)\Phi_2(t) - \ldots - 2c_n f(t)\Phi_n(t))dt \tag{12}$$

Let $\kappa_i = \langle \Phi_i | \Phi_i \rangle_\Gamma$, and $\gamma_i = \langle \overline{f} | \Phi_i \rangle_\Gamma$.

$$MSE = \frac{1}{t_2 - t_1} \left\{ \left(\int_{t_1}^{t_2} f^2(t) dt \right) + c_1^2 \kappa_1 + c_2^2 \kappa_2 + \ldots + c_n^2 \kappa_n - 2c_1 \gamma_1 - 2c_2 \gamma_2 - \ldots - 2c_n \gamma_n \right\} \tag{13}$$

We write $c_i^2 \kappa_i - 2c_i \gamma_i = \left(c_i \sqrt{\kappa_i} - \frac{\gamma_i}{\sqrt{\kappa_i}} \right)^2 - \frac{\gamma_i^2}{\kappa_i}$, so

$$MSE = \frac{1}{t_2 - t_1} \left\{ \int_{t_1}^{t_2} f^2(t) dt + \sum_{i=1}^{n} (c_i \sqrt{\kappa_i} - \frac{\gamma_i}{\sqrt{\kappa_i}})^2 - \sum_{i=1}^{n} \frac{\gamma_i^2}{\kappa_i} \right\} \tag{14}$$

The goal is to minimize the mean squared error, but the only term we have control over is c_i. So we set the only term involving $c_i = 0$.

$$c_i\sqrt{\kappa_i} - \frac{\gamma_i}{\sqrt{\kappa_i}} = 0 \rightarrow c_i = \frac{\gamma_i}{\kappa_i} \tag{15}$$

$$c_i = \frac{\langle \overline{f}|\Phi_i\rangle_\Gamma}{\langle \Phi_i|\Phi_i\rangle_\Gamma} \tag{16}$$

2.4 The Complex Exponentials as a Basis

Let $\Psi_n(t) = e^{in\omega t}$. Ψ is periodic with a period of $T = \frac{2\pi}{\omega}$.

$$\langle \Psi_n|\Psi_m\rangle_\Gamma = \int_{t_o}^{t_o+\frac{2\pi}{\omega}} e^{i\omega t(m-n)}\,dt \tag{17}$$

For $n = m$, this obviously turns out to equal the period T. but when $n \neq m$,

$$\langle \Psi_n|\Psi_m\rangle_\Gamma = \frac{1}{i(m-n)\omega} e^{i(m-n)\omega t}\Big|_{to}^{t_o+\frac{2\pi}{\omega}} \tag{18}$$

$$\langle \Psi_n|\Psi_m\rangle_\Gamma = \frac{1}{i(m-n)\omega} e^{i(m-n)\omega t_o} \underbrace{\left(e^{i2\pi(m-n)} - 1\right)}_{0} \tag{19}$$

$$\langle \Psi_n|\Psi_m\rangle_\Gamma = 0 \tag{20}$$

3 The Fourier Series

Our goal is to represent a continuous, *periodic* function as a sum of complex sinusoids. We already know that these functions are an acceptable basis!

Consider a periodic signal $x(t)$ with period $T = \frac{2\pi}{\omega}$. We can recover the original $x(t)$:

$$x(t) = \sum_k a_k e^{ik\omega t} = \langle \overline{a_k}|\Psi_k\rangle \tag{21}$$

Recall that

$$a_k = \frac{\langle \Psi_k|x\rangle_T}{\langle \Psi_k|\Phi_k\rangle_T} = \frac{1}{T}\langle \Psi_k|x\rangle_T \tag{22}$$

Equation **??** is called the *Synthesis Equation* and equation **??** is called the *Analysis Equation.* The a_k's are called the *Fourier Series Coefficients*, or the *Spectral Coefficients.*

Unfortunately, this only works for periodic functions because of the orthogonality on an interval

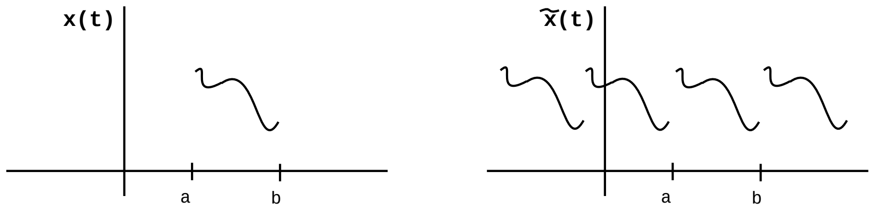

Figure 2: Aperiodic signal x(t) and a periodic version x̃(t)

4 Continuous Fourier Transforms

The basic idea is to approximate an aperiodic signal with a periodic one. We start with an aperiodic signal $x(t)$, where we are interested in some interval $[a, b]$.

We design a periodic function x̃(t) $= x(t)$ if $t \in [a, b]$. x̃(t) simply repeats the interval over and over.

The trick is to extend [a,b] to include the zero function on the left and right.

$$[a, b] \to [a - d, b + d]$$

$b - a = \omega$, and the period of x̃(t)$= T > \omega$ if $d \neq 0$.

The Fourier series coefficients for x̃ are:

$$a_k = \frac{1}{T}\langle \Psi_k | \tilde{x} \rangle_{[-\frac{T}{2}, \frac{T}{2}]} = \frac{1}{T}\langle \Psi_k | x \rangle_{[-\frac{T}{2}, \frac{T}{2}]} = \frac{1}{T}\langle \Psi_k | x \rangle = \frac{1}{T}\int x(t)e^{-ik\omega t}dt \tag{23}$$

These a_k are essentially scaled samples of a continuous function, $X(\omega)$.

$$X(\omega) = \int x(t)e^{-i\omega t}dt = \langle \Psi | x \rangle \tag{24}$$

So

$$a_k = \frac{1}{T}X(\omega k) \tag{25}$$

We have constructed this new periodic function and analyzed it to get these coefficients. What do they synthesize?

$$\tilde{x}(t) = \langle \overline{a_k} | \Psi \rangle \tag{26}$$

$$\tilde{x}(t) = \langle \frac{1}{T}\overline{X(\omega k} | \Psi \rangle \tag{27}$$

$$\tilde{x}(t) = \frac{1}{T}\langle \overline{X(\omega k)} | \Psi \rangle \tag{28}$$

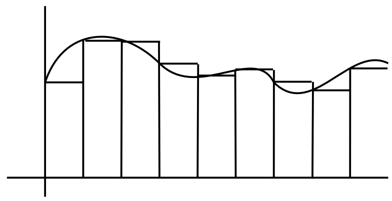

Figure 3: $X(\omega k)e^{ik\omega t}$

$$\tilde{x}(t) = \frac{1}{T}\sum_k x_c(\omega_k)e^{ik\omega t} \tag{29}$$

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_k x_c(\omega_k)e^{ik\omega t} \tag{30}$$

What is this summation?

Remember that we want $T \to \infty$ so this periodic function x̃(t) becomes $x(t)$. *But* as $T \to \infty$, $\omega \to 0$, so this summation simply becomes the integral!

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega \tag{31}$$

We now have the analysis and synthesis equations, **??** and **??**:

$$X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt = \frac{1}{2\pi}\langle\Psi|x\rangle \tag{32}$$

$$x(t) = \int_{-\infty}^{\infty} X(omega)e^{i\omega t}d\omega = \langle\Psi|x\rangle \tag{33}$$

X and x are called a *fourier pair.*

$$X(\omega) = \mathcal{F}\{x(t)\} \tag{34}$$

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} \tag{35}$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \tag{36}$$

4.1 Example: The box signal

Recall the box signal looks like Figure **??**.

$$X(\omega) = \int_{-\infty}^{\infty} Box(t)e^{-i\omega t}dt \tag{37}$$

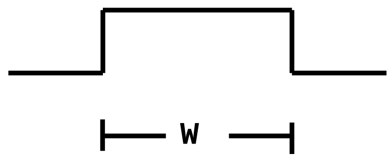

Figure 4: The Box signal

$$X(\omega) = \int_{-\frac{W}{2}}^{\frac{W}{2}} e^{-i\omega t} dt \tag{38}$$

$$X(\omega) = W sinc(\frac{W}{2\pi}\omega) \tag{39}$$

Does this make sense? We can prove it by synthesizing it!

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W sinc(\frac{W}{2\pi}\omega) e^{i\omega t} d\omega \tag{40}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W \frac{sin(\pi\frac{W}{2\pi}\omega)}{\pi\frac{W}{2\pi}\omega} e^{i\omega t} d\omega \tag{41}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{W} sin(\frac{W}{2}\omega) e^{i\omega t} \tag{42}$$

$$x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} [\cos\omega t + i\sin\omega t] \frac{sin(\frac{\omega W}{2})}{\omega} d\omega \tag{43}$$

$$x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \cos\omega t \frac{sin(\frac{\omega W}{2})}{\omega} d\omega + i\int_{-\infty}^{\infty} \sin\omega t \frac{sin(\frac{\omega W}{2})}{\omega} d\omega \tag{44}$$

The second term in this equation is an odd fcuntion, so the integral is zero! We can look up the solution to the first term in a table of integrals:

$$x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \cos\omega t \frac{sin(\frac{\omega W}{2})}{\omega} d\omega = \begin{cases} 1 & |t| < \frac{W}{2} \\ \frac{1}{2} & |t| = \frac{W}{2} \\ 0 & otherwise \end{cases} \tag{45}$$

Equation **??** is actually the box function! This verifies our original analysis equation.

We can also do this the other way around. Starting with the Box spectrum, we can analyze it and synthesize it to give a box.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Box(\omega) e^{i\omega t} d\omega \tag{46}$$

$$x(t) = \frac{1}{2\pi}\int_{-\frac{W}{2}}^{\frac{W}{2}} e^{i\omega t} d\omega \tag{47}$$

$$x(t) = \frac{1}{2\pi} W sinc(\frac{W}{2\pi} t) \tag{48}$$

The proof is the same except the zero term has a minus instead of a plus.

4.2 Example: Gaussians

An unnormalized gaussian can be represented as

$$g(t) = e^{-\frac{t^2}{\sigma^2}} \tag{49}$$

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{\sigma^2}} dt = \frac{1}{\sigma\sqrt{2\pi}} \tag{50}$$

The Fourier Transform is:

$$G(\omega) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{i\omega t} dt = \int_{-\infty}^{\infty} e^{-\pi t^2 - i\omega t} dt \tag{51}$$

By completing the square, we get:

$$G(\omega) = \int_{-\infty}^{\infty} e^{-\pi[t+(\frac{i\omega}{2\pi})^2] - \frac{\omega^2}{4\pi}} d\omega \tag{52}$$

$$G(\omega) = e^{-\frac{\omega^2}{2\pi}} \int_{-\infty}^{\infty} e^{-\pi\omega^2} d\omega \tag{53}$$

$$G(\omega) = e^{-\frac{\omega^2}{2\pi}} \tag{54}$$

This is a Gaussian!

4.3 Example: Delta Functions

$$x(t) = \delta(t) \tag{55}$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt \tag{56}$$

$$X(\omega) = e^0 \tag{57}$$

$$X(\omega) = 1 \tag{58}$$

So the fourier transform of a spike is a flat signal. This is known as a DC signal. The value of a fourier transform at $\omega = 0$ is called the DC component.

4.4 Example: Shah Functions

Consider a shah function with spikes at intervals of T apart, $\sum_k \delta(t - kT)$. The fourier coefficients are $\frac{1}{T}$, and the fourier transform is

$$X(\omega) = \frac{2\pi}{T} \sum_k \delta(\omega - k\frac{2\pi}{T}) \tag{59}$$

which is another shah function!

5 Fourier Transforms of Periodic Signals

Consider a simple example:

$$X(\omega) = 2\pi\delta(\omega - \omega_o) \tag{60}$$

The inverse fourier transform of this function is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega \tag{61}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_o) e^{i\omega t} d\omega \tag{62}$$

$$x(t) = e^{i\omega_o t} d\omega \tag{63}$$

This makes sense. $e^{i\omega_o t}$ has only one frequency, ω_o.

In general, we can consider some periodic signal x(t) with fourier series coefficients a_k.

$$X(\omega) = \sum_k 2\pi a_k \delta(\omega - k\omega_o) \tag{64}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\sum_k 2\pi a_k \delta(\omega - k\omega_o)) e^{i\omega t} d\omega \tag{65}$$

$$x(t) = \sum_k a_k \int \delta(\omega - k\omega_o) e^{i\omega t} d\omega \tag{66}$$

$$x(t) = \sum_k a_k e^{i\omega t} \tag{67}$$

6 The Convolution Theorem

What is the Fourier Transform of a convolved signal?

$$Y(\omega) = \int_{-\infty}^{\infty} f \otimes h e^{-i\omega t} dt \tag{68}$$

$$Y(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau e^{-i\omega t} dt \tag{69}$$

$$Y(\omega) = \int_{-\infty}^{\infty} f(\tau) \int_{-\infty}^{\infty} h(t-\tau) e^{-i\omega t} dt d\tau \tag{70}$$

$$Y(\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega t} H(\omega) d\tau \tag{71}$$

$$Y(\omega) = F(\omega) H(\omega) \tag{72}$$

So convolution in the time domain is multiplication in the frequency domain! We can similarly show the inverse:

$$f(t)h(t) = F(\omega) \otimes H(\omega) \tag{73}$$