Subdivision Surfaces

Jason Lawrence
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Acknowledgments: Denis Zorin, Peter Schröder, Tom Funkhouser
Subdivision Surfaces

- Coarse mesh & subdivision rule
- Smooth surface = limit of sequence of refinements

[Zorin & Schröder]
Two Key Questions

- How to refine mesh?
- Where to place new vertices?
  - Provable properties about limit surface

[Zorin & Schröder]
Loop Subdivision Scheme

- Each triangle => 4 new triangles:
  - split each edge, connect new vertices

[Zorin & Schröder]
Loop Subdivision Scheme

Choose locations for new vertices as weighted average of original vertices in local neighborhood
Loop Subdivision Scheme

Rules for extraordinary vertices and boundaries

a. Masks for odd vertices

b. Masks for even vertices

[Zorin & Schröder]
Loop Subdivision Scheme

- Choose $\beta$ by analyzing continuity of limit surface

- Original Loop

$$\beta = \frac{1}{n} \left( \frac{5}{8} - \left( \frac{3}{8} + \frac{1}{4} \cos \frac{2\pi}{n} \right)^2 \right)$$

- Warren

$$\beta = \begin{cases} \frac{3}{8n} & n > 3 \\ \frac{3}{16} & n = 3 \end{cases}$$
Analysis

- Limit surface has provable smoothness properties

[Zorin & Schröder]
Analysis

- Start with curve: 4-point interpolating scheme

(old points left where they are)
4-Point Scheme

What is the support?

Step i:

\[ \begin{align*}
\text{v}_{-2} & \quad \text{v}_{-1} & \quad \text{v}_0 & \quad \text{v}_1 & \quad \text{v}_2 \\
\end{align*} \]

Step i+1:

\[ \begin{align*}
\text{v}_{-2} & \quad \text{v}_{-1} & \quad \text{v}_0 & \quad \text{v}_1 & \quad \text{v}_2 \\
\end{align*} \]

So, 5 new points depend on 5 old points.
Subdivision Matrix

How are vertices in neighborhood refined?

\[
\begin{pmatrix}
\mathbf{v}_{-2}^{(i+1)} \\
\mathbf{v}_{-1}^{(i+1)} \\
\mathbf{v}_0^{(i+1)} \\
\mathbf{v}_1^{(i+1)} \\
\mathbf{v}_2^{(i+1)}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
-1/16 & 9/16 & 9/16 & -1/16 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -1/16 & 9/16 & 9/16 & -1/16 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{v}_{-2}^{(i)} \\
\mathbf{v}_{-1}^{(i)} \\
\mathbf{v}_0^{(i)} \\
\mathbf{v}_1^{(i)} \\
\mathbf{v}_2^{(i)}
\end{pmatrix}
\]
Subdivision Matrix

How are vertices in neighborhood refined?

\[
\begin{pmatrix}
  v_{-2}^{(i+1)} \\
v_{-1}^{(i+1)} \\
v_{0}^{(i+1)} \\
v_{1}^{(i+1)} \\
v_{2}^{(i+1)}
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 & 0 & 0 & 0 \\
-1/16 & 9/16 & 9/16 & -1/16 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -1/16 & 9/16 & 9/16 & -1/16 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
v_{-2}^{(i)} \\
v_{-1}^{(i)} \\
v_{0}^{(i)} \\
v_{1}^{(i)} \\
v_{2}^{(i)}
\end{pmatrix}
\]

\[v^{(i+1)} = Sv^{(i)}\]
Subdivision Matrix

How are vertices in neighborhood refined?

\[ V^{(i+1)} = S V^{(i)} \]

single round:

\[ V^{(n)} = S^n V^{(0)} \]

after n rounds:
Convergence Criterion

Expand in eigenvectors of $S$:

$$V^{(0)} = \sum_{i=0}^{4} \alpha_i e_i$$

$$V^{(n)} = \sum_{i=0}^{4} \alpha_i \lambda_i^n e_i$$

Criterion I: $|\lambda_i| \leq 1$
Convergence Criterion

- What if all eigenvalues of $S$ are $< 1$?
  - All points converge to 0 with repeated subdivision

Criterion 2: $\lambda_0 = 1$
Translation Invariance

For any translation $t$, we desire:

\[
\begin{pmatrix}
\vec{v}_{(i+1)} - 2 + t \\
\vec{v}_{(i+1)} - 1 + t \\
\vec{v}_{(i+1)} 0 + t \\
\vec{v}_{(i+1)} 1 + t \\
\vec{v}_{(i+1)} 2 + t \\
\end{pmatrix}
= S
\begin{pmatrix}
\vec{v}_{(i)} - 2 + t \\
\vec{v}_{(i)} - 1 + t \\
\vec{v}_{(i)} 0 + t \\
\vec{v}_{(i)} 1 + t \\
\vec{v}_{(i)} 2 + t \\
\end{pmatrix}
\]

\[
\vec{V}_{(i+1)} + t\vec{1} = S(\vec{V}(i) + t\vec{1})
\]

\[S\vec{1} = \vec{1}\]

Criterion 3: $e_0 = \vec{1}$, all other $|\lambda_i| < 1$
Smoothness Criterion

- Plug back in:
  
  $$V^{(n)} = a_0 e_0 + \sum_{i=1}^{4} a_i \lambda_i^n e_i$$

- Dominated by largest eigenvalue $\lambda_i$

- **Case 1**: $|\lambda_1| > |\lambda_2|$
  
  $$V^{(n)} = a_0 e_0 + a_0 \lambda_1^n e_1 + \epsilon$$

- Group of 5 points get closer to one another
- All points approach multiples of $e_1 \rightarrow$ lie along a straight line
- Smooth!!
Smoothness Criterion

Case 2: $|\lambda_1| = |\lambda_2|$

- Points can lie anywhere in the span of $(e_1, e_2)$
- No longer have smoothness guarantee

Criterion 4: Smooth iff $\lambda_0 = 1 > |\lambda_1| > |\lambda_i|$
Continuity and Smoothness

So, what about our 4-point scheme?

- eigenvalues = 1, 1/2, 1/4, 1/4, 1/8
- \( e_0 = \vec{1} \)
- stable ✓
- translation invariant ✓
- smooth ✓
2-Point Scheme

- In contrast, consider 2-point interpolating scheme

- Support = 3

- Subdivision matrix = \[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 1 & 0 \\
0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]
2-Point Scheme

- eigenvalues = 1, 1/2, 1/2
- $e_0 = \vec{1}$
- stable ✓
- translation invariant ✓
- smooth X
  - not smooth; in fact, this is piecewise linear
What About Surfaces?

- Similar analysis: determine support, construct subdivision matrix, find eigenstructure
- Caveat 1: separate analysis for each vertex valence
- Caveat 2: consider more than 1 subdominant eigenvalue

Reif’s smoothness condition: \( \lambda_0 = 1 > |\lambda_1| \geq |\lambda_2| > |\lambda_i| \)

- Points lie in subspace spanned by \((e_1, e_2)\)
- If \( |\lambda_1| \neq |\lambda_2| \), neighborhood stretched when subdivided, but remains 2-manifold surface
Limit Surfaces

- Behavior of surfaces determined by eigenvalues

- Recall that symmetric matrices have real eigenvalues
Butterfly Subdivision

- Interpolating subdivision: larger neighborhood
Modified Butterfly Scheme

- Need special weights near extraordinary vertices
  - For n=3, weights are 5/12, -1/12, -1/12
  - For n=4, weights are 3/8, 0, -1/8, 0
  - For n>=5, weights are

\[
\frac{1}{n} \left( \frac{1}{4} + \cos \frac{2\pi j}{n} + \frac{1}{2} \cos \frac{4\pi j}{n} \right), \quad j = 0 \ldots n - 1
\]
Variety of Schemes

- Triangles vs. Quads
- “Interpolating” vs. “Approximating”

<table>
<thead>
<tr>
<th>Face split</th>
<th>Triangular meshes</th>
<th>Quad. meshes</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Approximating</strong></td>
<td>Loop ($C^2$)</td>
<td>Catmull-Clark ($C^2$)</td>
</tr>
<tr>
<td><strong>Interpolating</strong></td>
<td>Mod. Butterfly ($C^1$)</td>
<td>Kobbelt ($C^1$)</td>
</tr>
</tbody>
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<tbody>
<tr>
<td>Doo-Sabin, Midedge ($C^1$)</td>
<td></td>
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<tr>
<td>Biquartic ($C^2$)</td>
<td></td>
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</tbody>
</table>

[Zorin & Schröder]
More Exotic Methods

- Kobbelt's subdivision:

  Number of faces triples per iteration: finer control over polygon count
Subdivision Schemes

- Loop
- Butterfly
- Catmull-Clark
- Doo-Sabin

[Zorin & Schröder]
Subdivision Schemes

Loop  Butterfly  Catmull-Clark  Doo-Sabin
Summary

Advantages:
- simple method for describing complex, smooth surfaces
- easy to implement
- arbitrary topology
- local support
- guaranteed continuity
- multiresolution

Difficulties:
- intuitive specification
- parameterization
- intersections