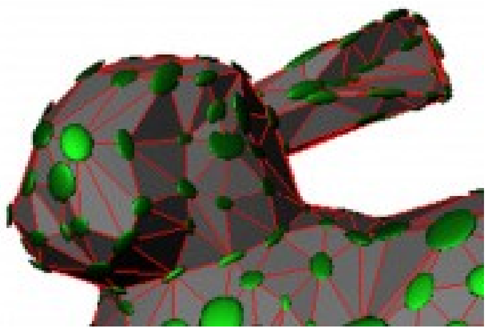


Pair contraction and the Quadric error metric.

Diego Nehab

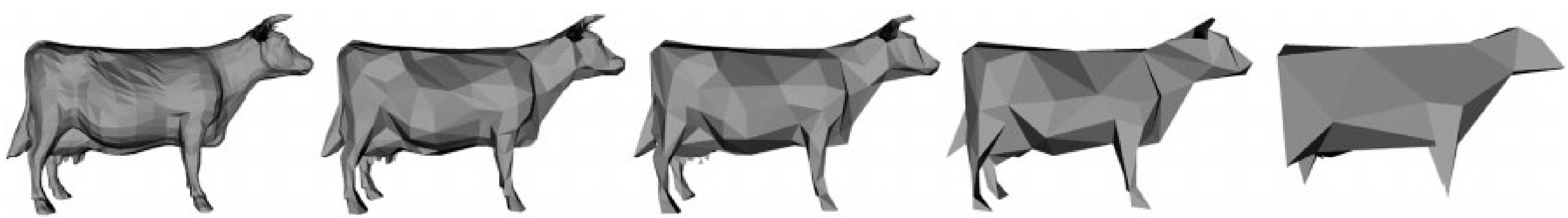
Pictures (and ideas) by Garland

Surface simplification: definition

- Starting with model M_n, generate sequence of simplified models M_{n-1}, M_{n-2}, $\cdots$, M_G
- To get from M_n to M_{n-1}, *contract* a pair of vertices:
  - $(\mathbf{v}_i, \mathbf{v}_j) \in \mathsf{M}_n \rightarrow \overline{\mathbf{v}} \in \mathsf{M}_{n-1}$
  - Move edges from $\mathbf{v}_i$ and $\mathbf{v}_j$ to $\overline{\mathbf{v}}$
  - Remove degenerate faces
- Resulting sequence M is a multiresolution representation for M_n: a progressive mesh.

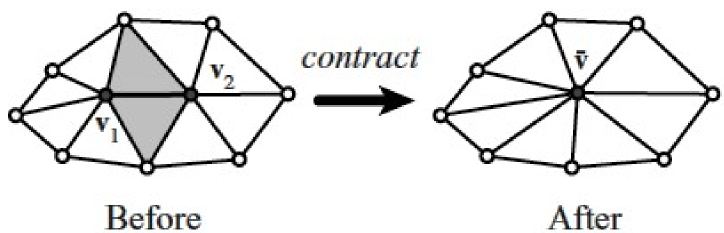
Surface simplification: Garland's goals

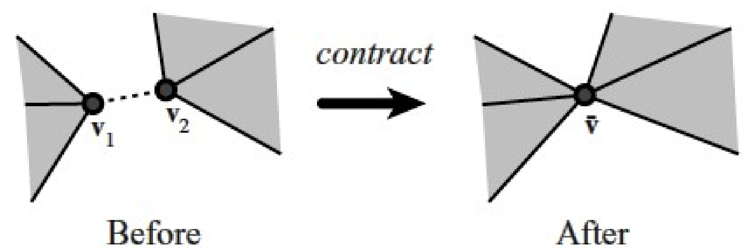
- Efficiency: 70.000 to 100 faces in 15s at SIGGRAPH'97
- Quality: preserve primary features, caring about appearance rather than maintaining topology
- Generality: support for non-manifold surfaces, collapsing disjoint regions when appropriate.

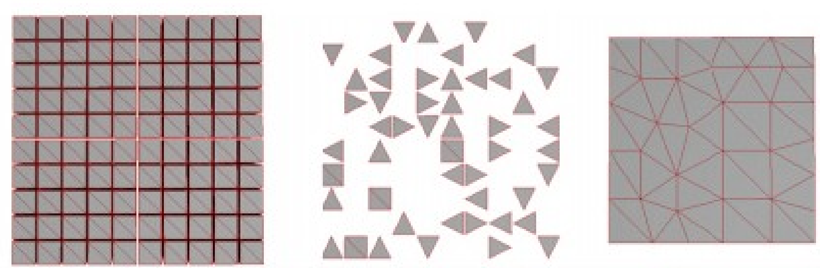


Pair contraction vs. edge contraction

- $(\mathbf{v}_i, \mathbf{v}_j)$ are not restricted to edges in M_n (for generality)
  - Previously disconnected regions may become connected after contraction
  - Less sensitivity to mesh connectivity (duplications are consolidated).







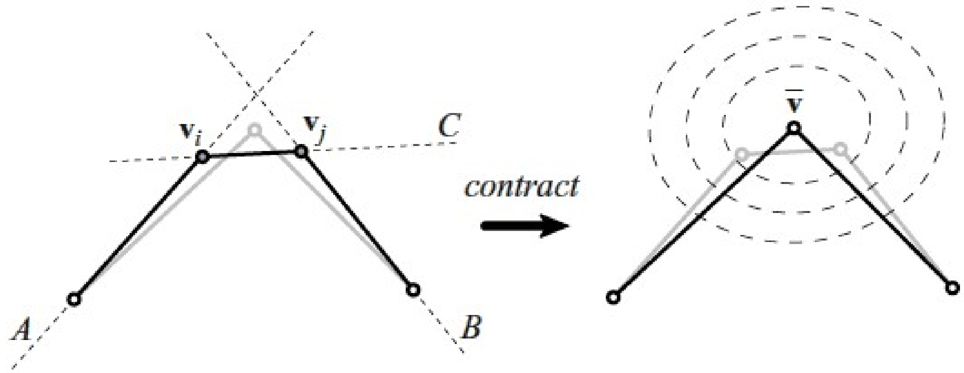
Outline of algorithm

- Restrict process to a set of *valid pairs*:
  - $(\mathbf{v}_i, \mathbf{v}_j)$ is an edge, or
  - $||\mathbf{v}_i - \mathbf{v}_j|| < t$, a threshold
    - $t = 0$ restricts to edge contraction
    - $t \gg 0$ can connect distant regions or yield $O(n^2)$ pairs
- Iteratively remove *best* pair and update valid pairs list:
  - Each vertex has a set with the pairs it belongs to:
    - $\mathbf{v}_i \rightarrow \text{Pairs}(\mathbf{v}_i)$
    - $(\mathbf{v}_i, \mathbf{v}_j) \rightarrow \overline{\mathbf{v}} \Rightarrow \text{Pairs}(\overline{\mathbf{v}}) = \text{Pairs}(\mathbf{v}_i) \cup \text{Pairs}(\mathbf{v}_j)$
- But how to choose *best* pair?

What error metric?

- So we need an error metric:
  - Must be inexpensive (for efficiency)
  - Must characterize approximation error (for quality)
- Will use plane-based error metric
- Each vertex $\mathbf{v}_i$ has a set P_i of faces incident to it, each face defines a plane π:
- Define the error $E_\pi(\mathbf{u}) = D^2_\pi(\mathbf{u})$
  - $\pi : \mathbf{n}^{\mathbf{t}}_\pi \mathbf{u} + d_\pi = 0, \qquad ||\mathbf{n}_\pi|| = 1$
  - $D^2\pi(\mathbf{u}) = (\mathbf{n}^{\mathbf{t}}_\pi \mathbf{u} + d_\pi)^2 = (ax + by + cz + d)^2$
- Define the error $E_{\mathbf{v}_i}(\mathbf{u}) = E_{P_i}(\mathbf{u}) = \sum_{P_i} D^2_\pi(\mathbf{u})$
  - $E_{\mathbf{v}_i}(\mathbf{u}) = \sum_{P_i} (\mathbf{n}^{\mathbf{t}}_\pi \mathbf{u} + d_\pi)^2$

2D point of view



- Notice:
  - $E_{\mathbf{v}_i}(\mathbf{v}_i) = E_{\mathbf{v}_j}(\mathbf{v}_j) = 0$
  - $P_i = \{A, C\}$, $P_j = \{C, B\}$, $\overline{P} = P_i \cup P_j = \{A, B, C\}$
  - $E(\overline{\mathbf{v}}) > 0$
  - Isocontours $E(\overline{\mathbf{v}}) = c$ give ellipses
  - $\min E(\overline{\mathbf{v}})$ is for $\overline{\mathbf{v}}$ at center.
- But do we need to keep track of all original planes?

Quadric error metric: definition

- Rewrite $D^2(\mathbf{v})$ to get:

$$\begin{aligned} D^2(\mathbf{v}) &= (\mathbf{n^t v} + d)^2 \\ &= (\mathbf{v^t n} + d)(\mathbf{n^t v} + d) \\ &= (\mathbf{v^t n n^t v} + 2d\mathbf{n^t v} + d^2) \\ &= \left(\mathbf{v^t (n n^t) v} + 2(d\mathbf{n^t})\mathbf{v} + d^2\right) \end{aligned}$$

- Define $Q = (\mathbf{A}, \mathbf{b}, c)$
- Define $Q(\mathbf{v}) = \mathbf{v^t A v} + 2\mathbf{b^t v} + c$
- For plane $\pi : \mathbf{n^t v} + d = 0$, let $Q_\pi(\mathbf{n n^t}, d\mathbf{n}, d^2)$ be its *fundamental quadric*.

Quadric error metric: properties

- $Q_\pi(\mathbf{v}) = D_\pi^2(\mathbf{v})$
- $Q_i(\mathbf{v}) + Q_j(\mathbf{v}) = (Q_i + Q_j)(\mathbf{v})$
  where $Q_i + Qj = (\mathbf{A}_i + \mathbf{A}_j, \mathbf{b}_i + \mathbf{b}_j, c_i + c_j)$
- $E_P(\mathbf{v}) = \sum_P D_\pi^2(\mathbf{v}) = \sum_P Q_\pi(\mathbf{v}) = (\sum_P Q_\pi)(\mathbf{v}) = Q_P(\mathbf{v})$
- But do we need to keep track of all original planes?
  - No! Keep one quadric per vertex
  - After contraction $(\mathbf{v}_i, \mathbf{v}_j) \rightarrow \overline{\mathbf{v}}$, we have $\overline{Q} = Q_i + Q_j$
  - Better yet, cost of contraction is $\overline{Q}(\overline{\mathbf{v}}) = Q_i(\overline{\mathbf{v}}) + Q_j(\overline{\mathbf{v}})$
- Isosurfaces $Q(\mathbf{v}) = c$ give ellipsoids, paraboloids, hyperboloids and planes, thence the name *quadric* error metric.

Final algorithm

- Compute Q_i for all vertices $\mathbf{v}_i$
- Determine valid pairs
- Compute optimal contraction target and associated quadric error for each pair
- Place all pairs in a heap, ordered by smallest error
- Repeat
  - Get least error pair $(\mathbf{v}_i, \mathbf{v}_j)$ from heap
  - Contract pair (move edges to $\overline{\mathbf{v}}$, remove degenerate planes)
  - Update cost for all pairs involving $\mathbf{v}_i$ and $\mathbf{v}_j$
- Until done.

Omitted details

- Computing optimum $\overline{\mathbf{v}}$ given Q
- Used data structures
- Weighting of quadrics
- Geometric interpretation of error

Bibliography

- Quadric-Based Polygonal Surface Simplification, Michael Garland, Ph.D. Thesys, Garnegie Mellon University, 1999
- Surface Simplification Using Quadric Error Metrics, Michael Garland and Paul Heckbert, SIGGRAPH'97