Parametric Curves

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CS4810: Introduction to Graphics

Acknowledgment: slides by Misha Kazhdan, Allison Klein, Tom Funkhouser, Adam Finkelstein and David Dobkin
Parametric Curves and Surfaces

Part 1: Curves

Part 2: Surfaces

Courtesy of C.K. Shene
Curves

• Splines: mathematical way to express curves

• Motivated by “loftsman’s spline”
  o Long, narrow strip of wood/plastic
  o Used to fit curves through specified data points
  o Shaped by lead weights called “ducks”
  o Gives curves that are “smooth” or “fair”

• Have been used to design:
  o Automobiles
  o Ship hulls
  o Aircraft fuselage/wing
Goals

• Some attributes we might like to have:
  - Predictable/local control
  - Simple
  - Continuous

• We’ll satisfy these goals using:
  - Piecewise
  - Polynomials
Many applications in graphics

• Animation paths

• Shape modeling

• etc…
What is a Spline in CG?

A spline is a *piecewise polynomial function* whose derivatives satisfy some *continuity constraints* across curve boundaries.

So let’s look at what this means…
What is a Spline in CG?

**Piecewise**: the spline is actually a collection of individual segments joined together.

**Polynomial functions**: each of these segments is expressed by a polynomial function.
**Parametric Curves**

A parametric curve in $d$-dimensions is defined by a collection of 1D functions of one variable that give the coordinates of points on the curve at each value of $u$:

$$\Phi(u) = \left(x_1(u), \ldots, x_d(u)\right)$$

\[ \Phi(u) = (\cos(u), u, \sin(u)) \]

**Note:**
A parametric curve is **not** the graph of a function, it is the path traced out as the value of $t$ is allowed to change.

Courtesy of C.K. Shene
**Derivatives**

If \( \Phi(u) = (x(u), y(u)) \) is the parametric equation of a curve, the parametric derivative of the curve at a point \( u_0 \) is the vector:

\[
\Phi'(u_0) = \left( x'(u_0), y'(u_0) \right)
\]

which points in a direction tangent to the curve.

Note:
The direction of the derivative is determined by the path that the curve traces out.

The magnitude of the parametric derivative is determined by the tracing speed.
Polynomials

A polynomial in the variable $u$ is:

• “An algebraic expression written as a sum of constants multiplied by different powers of a variable.”

$$P(u) = a_0 + a_1u + a_2u^2 + \ldots + a_nu^n = \sum_{k=0}^{n} a_k u^k$$

The constant $a_k$ is referred to as the $k$-th coefficient of the polynomial $P$. 
Polynomials (Degree)

\[ P(u) = a_0 + a_1 u + a_2 u^2 + \ldots + a_n u^n = \sum_{k=0}^{n} a_k u^k \]

A polynomial \( P \) has degree \( n \) if for all \( k > n \), the coefficients of the polynomial satisfy \( a_k = 0 \). 

Wednesday, November 16, 11
Polynomials (Degree)

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A polynomial \( P \) has degree \( n \) if for all \( k > n \), the coefficients of the polynomial satisfy \( a_k = 0 \).

A polynomial of degree \( n \) has \( n + 1 \) degrees of freedom.

Knowing \( n + 1 \) pieces of information about a polynomial of degree \( n \) gives enough information to reconstruct the coefficients.
Polynomials (Matrices)

\[ P(u) = a_0 + a_1 u + a_2 u^2 + \ldots + a_n u^n = \sum_{k=0}^{n} a_k u^k \]

The polynomial \( P \) can be expressed as the matrix multiplication of a column vector and a row vector:

\[ P(u) = \begin{pmatrix} u^n & \ldots & u^0 \end{pmatrix} \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix} \]
Polynomials (Matrices)

$P(u) = \sum_{k=0}^{n} a_k u^k$

Example:

If we know the values of the polynomial $P$ at $n+1$ different values:

$$P(u_0) = p_0, \ldots, P(u_n) = p_n$$

We can compute the coefficients of $P$ by inverting the appropriate matrix:

$$\begin{pmatrix} p_0 \\
\vdots \\
p_n \end{pmatrix} = \begin{pmatrix} (u_0)^n & \cdots & (u_0)^0 \\
\vdots & \ddots & \vdots \\
(u_n)^n & \cdots & (u_n)^0 \end{pmatrix} \begin{pmatrix} a_n \\
\vdots \\
a_0 \end{pmatrix}$$

$$\begin{pmatrix} a_n \\
\vdots \\
a_0 \end{pmatrix} = \begin{pmatrix} (u_0)^n & \cdots & (u_0)^0 \end{pmatrix}^{-1} \begin{pmatrix} p_0 \\
\vdots \\
p_n \end{pmatrix}$$
Polynomials (Matrices)

Example:

So, if we are given the values of the polynomial $P$ at the $n+1$ positions $u_0, \ldots, u_n$, we can compute the value of $P$ at any position $u$ by solving:

$$P(u) = \left( u^n \ldots u^0 \right) \begin{pmatrix} (u_0)^n \cdots (u_0)^0 \end{pmatrix}^{-1} \begin{pmatrix} p_0 \\ \vdots \\ (u_n)^n \cdots (u_n)^0 \end{pmatrix} \begin{pmatrix} p_0 \\ \vdots \\ p_n \end{pmatrix}$$
Parametric Polynomial Curves

• A parametric polynomial curve of degree $n$ in $d$ dimensions is a collection of $d$ polynomials, each of which is of degree no larger than $n$:

$$\Phi(u) = \left( x_1(u) = \sum_{k=0}^{n} a_{1,k} u^k, \ldots, x_d(u) = \sum_{k=0}^{n} a_{d,k} u^k \right)$$
Parametric Polynomial Curves

Examples:

- When $x(u)=u$, the curve is just the graph of $y(u)$.
- Different parametric equations can trace out the same curve.
- As the degree gets larger, the complexity of the curve increases.

<table>
<thead>
<tr>
<th>$y(u)=u$</th>
<th>$y(u)=u^2/2-2$</th>
<th>$y(u)=u^3/2-2u$</th>
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Parametric Curves

Goal:

Given a collection of \( m \) points in \( d \) dimensions:

\[
\{ p_1 = (x_{1,1}, \ldots, x_{1,d}) \ldots, p_m = (x_{m,1}, \ldots, x_{m,d}) \}\]

define a parametric curve that passes through (or near) the points.
Parametric Curves

Direct Approach:

Solve for the $m$ coefficients of a parametric polynomial curve of degree $m-1$, passing through the points.
Parametric Curves

Direct Approach:

Solve for the $m$ coefficients of a parametric polynomial curve of degree $m-1$, passing through the points.

Limitations:

• No local control

• As the number of points increases, the dimension gets larger, and the curve oscillates more.
Splines

Approach:

Fit low-order polynomials to groups of points so that the combined curve passes through (or near) the points while providing:

- Local Control
- Simplicity
- Continuity/Smoothness
Splines

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**Piecewise parametric polynomials**

**Approach:**

Fit low-order polynomials to groups of points so that the combined curve passes through (or near) the points while providing:

- **Local Control:**
  - Individual curve segments are defined using only local information

- **Simplicity**
  - Curve segments are low-order polynomials
Approach:

Fit low-order polynomials to groups of points so that the combined curve passes through (or near) the points while providing:

- **Local Control:**
  - Individual curve segments are defined using only local information

- **Simplicity**
  - Curve segments are low-order polynomials

- **Continuity/Smoothness**
  - How do we guarantee smoothness at the joints?
Continuity/Smoothness

Continuity:

Within the parameterized domain, the polynomial functions are continuous and smooth.

The derivatives of our polynomial functions must satisfy continuity constraints across the curve boundaries.
Continuity/Smoothness

Parametric continuity: derivatives of the two curves are equal where they meet.

- $C^0$ means two curves just meet
- $C^1$ means 1$^{st}$ derivatives equal
- $C^2$ means both 1$^{st}$ and 2$^{nd}$ derivates equal
Continuity/Smoothness

Geometric continuity: derivatives of the two curves are proportional (i.e. point in the same direction) where they meet.

- $G^0$ means two curves just meet
- $G^1$ means $G^0$ and 1$^{\text{st}}$ derivatives proportional
- $G^2$ means $G^1$ and 2$^{\text{nd}}$ derivatives proportional
- Parametric continuity used more frequently than geometric.
What is a Spline in CG?

A spline is a **piecewise polynomial function** whose derivatives satisfy some **continuity constraints** across curve boundaries.

$$P_i(x) \quad x \in [0,1).$$
What is a Spline in CG?

A spline is a piecewise polynomial function whose derivatives satisfy some continuity constraints across curve boundaries.

$$P_i(x) = \sum_{j=0}^{n} a_{ij}x^j$$
Overview

• What is a Spline?

• Specific Examples:
  o Hermite Splines
  o Cardinal Splines
  o Uniform Cubic B-Splines

• Comparing Cardinal Splines to Uniform Cubic B-Splines
Specific Example: Hermite Splines

- Interpolating piecewise *cubic* polynomial

- Specified with:
  - A pair of control points
  - Tangent at each control point

- Iteratively construct the curve between adjacent end points
Specific Example: Hermite Splines

• Interpolating piecewise cubic polynomial

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Specific Example: Hermite Splines

- Interpolating piecewise *cubic* polynomial

- Specified with:
  - A pair of control points
  - Tangent at each control point

- Iteratively construct the curve between adjacent end points

Because the end-points of adjacent curves share the same position and derivatives, the Hermite spline has $C^1$ continuity.
Specific Example: Hermite Splines

- Let $P_k(u) = (P_{k,x}(u), P_{k,y}(u))$ with $0 \leq u \leq 1$ be a parametric cubic point function for the curve section between control points $p_k$ and $p_{k+1}$.

- Boundary conditions are:
  - $P_k(0) = p_k$
  - $P_k(1) = p_{k+1}$
  - $P'_k(0) = Dp_k$
  - $P'_k(1) = Dp_{k+1}$
Specific Example: Hermite Splines

Let $P_k(u) = (P_{k,X}(u), P_{k,Y}(u))$ with $0 \leq u \leq 1$ be a parametric cubic point function for the curve section between control points $p_k$ and $p_{k+1}$.

- Boundary conditions are:
  - $P_k(0) = p_k$
  - $P_k(1) = p_{k+1}$
  - $P_k'(0) = Dp_k$
  - $P_k'(1) = Dp_{k+1}$

- Solve for the coefficients of the polynomials $P_{k,X}(u)$ and $P_{k,Y}(u)$ that satisfy the boundary condition.
Specific Example: Hermite Splines

We can express the polynomials:
• \( P(u) = au^3 + bu^2 + cu + d \)
• \( P'(u) = 3au^2 + 2bu + c \)

using the matrix representations:

\[
\begin{align*}
P(u) &= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\
\end{align*}
\]

\[
\begin{align*}
P'(u) &= \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\
\end{align*}
\]
Specific Example: Hermite Splines

We can express the polynomials:

- \( P(u) = au^3 + bu^2 + cu + d \)
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using the matrix representations:

\[
\begin{bmatrix}
    u^3 & u^2 & u & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix}
\quad
\begin{bmatrix}
    3u^2 & 2u & 1 & 0
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix}
\]

By abuse of notation, we will think of the coefficients \(a, b, c,\) and \(d\) as 2-vectors rather than scalars so that \(P\) is a function taking values in 2D.
Specific Example: Hermite Splines

Given the matrix representations:

\[ P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \\ a \\ b \\ c \\ d \end{bmatrix} \quad P'(u) = \begin{bmatrix} u^2 & 2u & 1 & 0 \end{bmatrix} \]
Specific Example: Hermite Splines

Given the matrix representations:

\[
P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}
\]

\[
P'(u) = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}
\]

we can express the values at the end-points as:

\[
p_k = P(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}
\]

\[
Dp_k = P'(0) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}
\]

\[
p_{k+1} = P(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}
\]

\[
Dp_{k+1} = P'(1) = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}
\]
Specific Example: Hermite Splines

We can combine the equations

\[ p_k = P(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ a \\ b \\ c \\ d \end{bmatrix} \quad Dp_k = P'(0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ a \\ b \\ c \\ d \end{bmatrix} \]

\[ p_{k+1} = P(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a \\ b \\ c \\ d \end{bmatrix} \quad Dp_{k+1} = P'(1) = \begin{bmatrix} 3 & 2 & 1 & 0 \\ a \\ b \\ c \\ d \end{bmatrix} \]

into a single matrix expression:
Specific Example: Hermite Splines

We can combine the equations

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p_k = P(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ a \\ b \\ c \\ d \end{bmatrix}, \quad Dp_k = P'(0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ a \\ b \\ c \\ d \end{bmatrix}
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\]

into a single matrix expression:

\[
\begin{bmatrix}
  p_k \\
  p_{k+1} \\
  Dp_k \\
  Dp_{k+1}
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 & 0 & 1 \\
  1 & 1 & 1 & 1 \\
  0 & 0 & 1 & 0 \\
  3 & 2 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  a \\
  b \\
  c \\
  d
\end{bmatrix}
\]
Specific Example: Hermite Splines

Inverting the matrix in the equation:

\[
\begin{bmatrix}
    p_k \\
p_{k+1} \\
    Dp_k \\
    Dp_{k+1}
\end{bmatrix} = \begin{bmatrix}
    0 & 0 & 0 & 1 \\
    1 & 1 & 1 & 1 \\
    0 & 0 & 1 & 0 \\
    3 & 2 & 1 & 0
\end{bmatrix}\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix}
\]

we get:
Specific Example: Hermite Splines

Inverting the matrix in the equation:

\[
\begin{bmatrix}
p_k \\
p_{k+1} \\
Dp_k \\
Dp_{k+1}
\end{bmatrix} = \begin{bmatrix}
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0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0
\end{bmatrix} \begin{bmatrix}
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b \\
c \\
d
\end{bmatrix}
\]

we get:

\[
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
3 & 2 & 1 & 0
\end{bmatrix}^{-1} \begin{bmatrix}
p_k \\
p_{k+1} \\
Dp_k \\
Dp_{k+1}
\end{bmatrix}
\]
Specific Example: Hermite Splines

Inverting the matrix in the equation:

\[
\begin{bmatrix}
  p_k \\
  p_{k+1} \\
  Dp_k \\
  Dp_{k+1}
\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}
\]

we get:

\[
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix}
\]
Specific Example: Hermite Splines

Using the facts that:

\[
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix} =
\begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
p_k \\
p_{k+1} \\
Dp_k \\
Dp_{k+1}
\end{bmatrix}
\]

we get:

\[
P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}
\]
Specific Example: Hermite Splines

Using the facts that:

\[
\begin{bmatrix}
  a \\
  b \\
  c \\
  d
\end{bmatrix}
= \begin{bmatrix}
  2 & -2 & 1 & 1 \\
  -3 & 3 & -2 & -1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  p_k \\
  p_{k+1} \\
  Dp_k \\
  Dp_{k+1}
\end{bmatrix}
\]

we get:

\[
P(u) = \begin{bmatrix}
  u^3 & u^2 & u & 1
\end{bmatrix}
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  2 & -2 & 1 & 1 \\
  -3 & 3 & -2 & -1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  a \\
  b \\
  c \\
  d
\end{bmatrix}
\]

parameters \( M_{\text{Hermite}} \) boundary info
Specific Example: Hermite Splines

and we can execute matrix multiplies below

\[ P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix} \]

to get

\[ P(u) = p_k (2u^3 - 3u^2 + 1) + p_{k+1} (-2u^3 + 3u^2) + \\
Dp_k (u^3 - 2u^2 + u) + Dp_{k+1} (u^3 - u^2) \]
Specific Example: Hermite Splines

Setting:

- \( H_0(u) = 2u^3 - 3u^2 + 1 \)
- \( H_1(u) = -2u^3 + 3u^2 \)
- \( H_2(u) = u^3 - 2u^2 + u \)
- \( H_3(u) = u^3 - u^2 \)

we can re-write the equation:

\[
P(u) = p_k (2u^3 - 3u^2 + 1) + p_{k+1} (-2u^3 + 3u^2) + \]

as:

\[
Dp_k (u^3 - 2u^2 + u) + Dp_{k+1} (u^3 - u^2) \]

\[
P(u) = p_k H_0(u) + p_{k+1} H_1(u) + Dp_k H_2(u) + Dp_{k+1} H_3(u) \]
Specific Example: Hermite Splines

Setting:

\[ H_0(u) = 2u^3 - 3u^2 + 1 \]
\[ H_1(u) = -2u^3 + 3u^2 \]
\[ H_2(u) = u^3 - 2u^2 + u \]
\[ H_3(u) = u^3 - u^2 \]

Blending Functions

\[ P(u) = p_k H_0(u) + p_{k+1} H_1(u) + Dp_k H_2(u) + Dp_{k+1} H_3(u) \]
Specific Example: Hermite Splines

Setting:
- $H_0(u) = 2u^3 - 3u^2 + 1$
- $H_1(u) = -2u^3 + 3u^2$
- $H_2(u) = u^3 - 2u^2 + u$
- $H_3(u) = u^3 - u^2$

When $u=0$:
- $H_0(u) = 1$
- $H_1(u) = 0$
- $H_2(u) = 0$
- $H_3(u) = 0$

So $P(0) = p_k$

$P(u) = p_k H_0(u) + p_{k+1} H_1(u) + Dp_k H_2(u) + Dp_{k+1} H_3(u)$
Specific Example: Hermite Splines

Setting:
- \( H_0(u) = 2u^3 - 3u^2 + 1 \)
- \( H_1(u) = -2u^3 + 3u^2 \)
- \( H_2(u) = u^3 - 2u^2 + u \)
- \( H_3(u) = u^3 - u^2 \)

When \( u = 1 \):
- \( H_0(u) = 0 \)
- \( H_1(u) = 1 \)
- \( H_2(u) = 0 \)
- \( H_3(u) = 0 \)

So \( P(1) = p_{k+1} \)

\[
P(u) = p_k H_0(u) + p_{k+1} H_1(u) + Dp_k H_2(u) + Dp_{k+1} H_3(u)
\]
Specific Example: Hermite Splines

Setting:

- $H_0(u) = 2u^3 - 3u^2 + 1$
- $H_1(u) = -2u^3 + 3u^2$
- $H_2(u) = u^3 - 2u^2 + u$
- $H_3(u) = u^3 - u^2$

When $u = 0$:

- $H_0'(u) = 0$
- $H_1'(u) = 0$
- $H_2'(u) = 1$
- $H_3'(u) = 0$

So $P'(0) = Dp_k$

$$P'(u) = p_k H_0'(u) + p_{k+1} H_1'(u) + Dp_k H_2'(u) + Dp_{k+1} H_3'(u)$$
Specific Example: Hermite Splines

Setting:
- \( H_0(u) = 2u^3 - 3u^2 + 1 \)
- \( H_1(u) = -2u^3 + 3u^2 \)
- \( H_2(u) = u^3 - 2u^2 + u \)
- \( H_3(u) = u^3 - u^2 \)

When \( u = 1 \):
- \( H_0'(u) = 0 \)
- \( H_1'(u) = 0 \)
- \( H_2'(u) = 0 \)
- \( H_3'(u) = 1 \)

So \( P'(1) = Dp_{k+1} \)

\[
P'(u) = p_k H_0'(u) + p_{k+1} H_1'(u) + Dp_k H_2'(u) + Dp_{k+1} H_3'(u)
\]
Specific Example: Hermite Splines

- Interpolating piecewise cubic polynomial

- Specified with:
  - Set of control points
  - Tangent at each control point

- Iteratively construct the curve between adjacent end points

Given the control points, how do we define the value of the tangents/derivatives?
Overview

• What is a Spline?

• Specific Examples:
  o Hermite Splines
  o Cardinal Splines
  o Uniform Cubic B-Splines

• Comparing Cardinal Splines to Uniform Cubic B-Splines
Specific Example: Cardinal Splines

- Interpolating piecewise *cubic* polynomial
- Specified with four control points
- Iteratively construct the curve between middle two points using adjacent points to define tangents
Specific Example: Cardinal Splines

• Interpolating piecewise cubic polynomial
• Specified with four control points
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Specific Example: Cardinal Splines

- Interpolating piecewise \emph{cubic} polynomial
- Specified with four control points
- Iteratively construct the curve between middle two points using adjacent points to define tangents

\[ p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7 \]
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Specific Example: Cardinal Splines

![Diagram showing cardinal splines with control points labeled as p₀, p₁, p₂, p₃, p₄, p₅, p₆, and p₇.](image-url)
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Because the end-points of adjacent curves share the same position and derivatives, the Cardinal spline has $C^1$ continuity.
Specific Example: Cardinal Splines

- Let $P_k(u) = (P_{k,X}(u), P_{k,Y}(u))$ with $0 \leq u \leq 1$ be a parametric cubic point function for the curve section between control points $p_k$ and $p_{k+1}$.

- Boundary conditions are:
  
  $\begin{align*}
  &\circ P(0) = p_k \\
  &\circ P(1) = p_{k+1} \\
  &\circ P'(0) = \frac{1}{2}(1 - t)(p_{k+1} - p_{k-1}) \\
  &\circ P'(1) = \frac{1}{2}(1 - t)(p_{k+2} - p_k)
  \end{align*}$

- Solve for the coefficients of the polynomials $P_{k,X}(u)$ and $P_{k,Y}(u)$ that satisfy the boundary condition.

Specific Example: Cardinal Splines
Specific Example: Cardinal Splines

Recall:

The Hermite matrix determines the coefficients of the polynomial from the positions and the derivatives of the end-points.

\[
\begin{bmatrix}
  p_k \\
p_{k+1}
\end{bmatrix} =
\begin{bmatrix}
  2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  u^3 \\
  u^2 \\
  u \\
  1
\end{bmatrix}
\]

where \(M_{\text{Hermite}}\) is the Hermite matrix and the parameters are \(u^3, u^2, u, 1\).
Specific Example: Cardinal Splines

Using same methods as with Hermite spline, from boundary conditions on previous slide we can get

\[
P(u) = \left[u^3 \quad u^2 \quad u \quad 1\right] \begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
p_k \\
p_{k+1} \\
s(p_{k+1} - p_{k-1}) \\
s(p_{k+2} - p_k)
\end{bmatrix}
\]

where \( s = (1 - t)/2 \)

The parameter \( t \) is called the **tension parameter**.
- Controls looseness versus tightness of curve.
Specific Example: Cardinal Splines

We can express the boundary conditions as a matrix applied to the points \( p_{k-1}, p_k, p_{k+1}, \) and \( p_{k+2} \):

\[
\begin{bmatrix}
    p_k \\
    p_{k+1} \\
    s(p_{k+1} - p_{k-1}) \\
    s(p_{k+2} - p_k)
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    -s & 0 & s & 0 \\
    0 & -s & 0 & s
\end{bmatrix}
\begin{bmatrix}
    p_{k-1} \\
    p_k \\
    p_{k+1} \\
    p_{k+2}
\end{bmatrix}
\]

to get
Specific Example: Cardinal Splines

We can express the boundary conditions as a matrix applied to the points $p_{k-1}$, $p_k$, $p_{k+1}$, and $p_{k+2}$:

$$
\begin{bmatrix}
  p_k \\
p_{k+1} \\
s(p_{k+1} - p_{k-1}) \\
s(p_{k+2} - p_k)
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  -s & 0 & s & 0 \\
  0 & -s & 0 & s
\end{bmatrix}
\begin{bmatrix}
p_{k-1} \\
p_k \\
p_{k+1} \\
p_{k+2}
\end{bmatrix}
$$

to get

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}
\begin{bmatrix}
  2 & -2 & 1 & 1 \\
  -3 & 3 & -2 & -1 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  -s & 0 & s & 0 \\
  0 & -s & 0 & s
\end{bmatrix}
\begin{bmatrix}
p_{k-1} \\
p_k \\
p_{k+1} \\
p_{k+2}
\end{bmatrix}$$
Specific Example: Cardinal Splines

Multiplying the interior matrices in:

\[
P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 0 & -s & 0 & s \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}
\]

we get the Cardinal matrix representation.
Specific Example: Cardinal Splines

Combining the matrices in:

\[
P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 0 & -s & 0 & s \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}
\]

we get the Cardinal matrix representation

\[
P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -s & 2 - s & s - 2 & s \\ 2s & s - 3 & 3 - 2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}
\]

\[M_{\text{Cardinal}}\]
Specific Example: Cardinal Splines

Setting:

- \( C_0(u) = -su^3 + 2su^2 - su \)
- \( C_1(u) = (2-s)u^3 + (s-3)u^2 + 1 \)
- \( C_2(u) = (s-2)u^3 + (3-2s)u^2 + su \)
- \( C_3(u) = su^3 - su^2 \)

Blending Functions

For \( s=0 \):

\[
P(u) = C_0(u)p_{k-1} + C_1(u)p_k + C_2(u)p_{k+1} + C_3(u)p_{k+2}
\]
Specific Example: Cardinal Splines

Setting:
- \( C_0(u) = -su^3 + 2su^2 - su \)
- \( C_1(u) = (2-s)u^3 + (s-3)u^2 + 1 \)
- \( C_2(u) = (s-2)u^3 + (3-2s)u^2 + su \)
- \( C_3(u) = su^3 - su^2 \)

For \( s=0 \):

Properties:
- \( C_0(u) + C_1(u) + C_2(u) + C_3(u) = 1 \)
- \( C_j(u) = C_{3-j}(1-u) \)
- \( C_0(1) = C_3(0) = 0 \)

\[
P(u) = C_0(u)p_{k-1} + C_1(u)p_k + C_2(u)p_{k+1} + C_3(u)p_{k+2}
\]
Specific Example: Cardinal Splines

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![Diagram of Cardinal Splines]

- $p_0$
- $p_1$
- $p_2$
- $p_3$
- $p_4$
- $p_5$
- $p_6$
- $p_7$
Specific Example: Cardinal Splines

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At the first and last end-points, you can:
- Not draw the final segments
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Specific Example: Uniform Cubic B-Splines

- Let $P_k(u) = (P_{k,X}(u), P_{k,Y}(u))$ with $0 \leq u \leq 1$ be a parametric cubic point function for the curve section around the control points $p_k$ and $p_{k+1}$.

- Boundary conditions are:
  - $P(0) = \frac{1}{6}(p_{k-1} + 4p_k + p_{k+1})$
  - $P(1) = \frac{1}{6}(p_k + 4p_{k+1} + p_{k+2})$
  - $P'(0) = \frac{1}{2}(1 - t)(p_{k+1} - p_{k-1})$
  - $P'(1) = \frac{1}{2}(1 - t)(p_{k+2} - p_k)$

- Solve for the coefficients of the polynomials $P_{k,X}(u)$ and $P_{k,Y}(u)$ that satisfy the boundary condition.
Specific Example: Uniform Cubic B-Splines

Using same methods as with Hermite spline, from boundary conditions on previous slide we can get

\[ P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{6} \\ p_{k-1} + 4p_k + p_{k+1} \\ p_k + 4p_{k+1} + p_{k+2} \\ 3p_{k+1} - 3p_{k-1} \\ 3p_{k+2} - 3p_k \end{bmatrix} \]

\[ M_{\text{Hermite}} \]
Specific Example: Uniform Cubic B-Splines

We can express the boundary conditions as a matrix applied to the points $p_{k-1}$, $p_k$, $p_{k+1}$, and $p_{k+2}$:

\[
\begin{bmatrix}
    p_{k-1} + 4p_k + p_{k+1} \\
    p_k + 4p_{k+1} + p_{k+2} \\
    3p_{k+1} - 3p_{k-1} \\
    3p_{k+2} - 3p_k
\end{bmatrix}
= \begin{bmatrix}
    1 & 4 & 1 & 0 \\
    0 & 1 & 4 & 1 \\
    -3 & 0 & 3 & 0 \\
    0 & -3 & 0 & 3
\end{bmatrix}
\begin{bmatrix}
    p_{k-1} \\
    p_k \\
    p_{k+1} \\
    p_{k+2}
\end{bmatrix}
\]

to get

\[
P(u) = \frac{1}{6} \begin{bmatrix}
    2 & -2 & 1 & 1 \\
    -3 & 3 & -2 & -1 \\
    0 & 0 & 1 & 0 \\
    1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    1 & 4 & 1 & 0 \\
    0 & 1 & 4 & 1 \\
    -3 & 0 & 3 & 0 \\
    0 & -3 & 0 & 3
\end{bmatrix}
\begin{bmatrix}
    p_{k-1} \\
    p_k \\
    p_{k+1} \\
    p_{k+2}
\end{bmatrix}
\]
Specific Example: Uniform Cubic B-Splines

Multiplying the interior matrices in:

\[
\begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 4 & 1 & 0 \\
0 & 1 & 4 & 1 \\
-3 & 0 & 3 & 0 \\
0 & -3 & 0 & 3
\end{bmatrix}
\begin{bmatrix}
p_{k-1} \\
p_k \\
p_{k+1} \\
p_{k+2}
\end{bmatrix}
\]

we get the cubic B-spline matrix representation
Specific Example: Uniform Cubic B-Splines

Combining the matrices in:

\[
P(u) = \frac{1}{6} \begin{bmatrix}
u^3 & u^2 & u & 1
\end{bmatrix} \begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 4 & 1 & 0 \\
0 & 1 & 4 & 1 \\
-3 & 0 & 3 & 0 \\
0 & -3 & 0 & 3
\end{bmatrix} \begin{bmatrix}
p_{k-1} \\
p_k \\
p_{k+1} \\
p_{k+2}
\end{bmatrix}
\]

we get the cubic B-spline matrix representation

\[
P(u) = \frac{1}{6} \begin{bmatrix}
u^3 & u^2 & u & 1
\end{bmatrix} \begin{bmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 0 & 3 & 0 \\
1 & 4 & 1 & 0
\end{bmatrix} \begin{bmatrix}
p_{k-1} \\
p_k \\
p_{k+1} \\
p_{k+2}
\end{bmatrix}
\]

\[
M_{\text{BSpline}}
\]
Specific Example: Uniform Cubic B-Splines

Setting:

- \( B_{0,3}(u) = 1/6(1-u)^3 \)
- \( B_{1,3}(u) = 1/6(3u^3-6u^2+4) \)
- \( B_{2,3}(u) = 1/6(-3u^3+3u^2+3u+1) \)
- \( B_{3,3}(u) = 1/6(u^3) \)

Blending Functions

\[
P(u) = B_{0,3}(u)p_{k-1} + B_{1,3}(u)p_k + B_{2,3}(u)p_{k+1} + B_{3,3}(u)p_{k+2}
\]
Specific Example: Uniform Cubic B-Splines

Setting:
- \( B_{0,3}(u) = \frac{1}{6}(1-u)^3 \)
- \( B_{1,3}(u) = \frac{1}{6}(3u^3 - 6u^2 + 4) \)
- \( B_{2,3}(u) = \frac{1}{6}(-3u^3 + 3u^2 + 3u + 1) \)
- \( B_{3,3}(u) = \frac{1}{6}(u^3) \)

Properties:
- \( B_{0,3}(u) + B_{1,3}(u) + B_{2,3}(u) + B_{3,3}(u) = 1 \)
- \( B_j(u) = B_{3-j}(1-u) \)
- \( B_{0,3}(1) = B_{3,3}(0) = 0 \)
- \( B_{j,3}(u) \geq 0 \)

\[
P(u) = B_{0,3}(u)p_{k-1} + B_{1,3}(u)p_k + B_{2,3}(u)p_{k+1} + B_{3,3}(u)p_{k+2}
\]
Specific Example: Uniform Cubic B-Splines

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Blending Functions

Blending functions provide a way for expressing the functions $P_k(u)$ as a weighted sum of the four control points $p_{k-1}$, $p_k$, $p_{k+1}$, and $p_{k+2}$:

$$P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2}$$
Blending Functions

Properties:

• Translation Commutativity:
  
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1, \text{ for all } 0 \leq u \leq 1. \]
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1, \text{ for all } 0 \leq u \leq 1. \]
  If we translate all the control points by the same vector \( q \), the position of the new point at the value \( u \) will just be the position of the old value at \( u \), translated by \( q \):
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1, \] for all \( 0 \leq u \leq 1. \]
  If we translate all the control points by the same vector \( q \), the position of the new point at the value \( u \) will just be the position of the old value at \( u \), translated by \( q \):

\[
Q_k(u) = BF_0(u)(q + p_{k-1}) + BF_1(u)(q + p_k) + BF_2(u)(q + p_{k+1}) + BF_3(u)(q + p_{k+2})
\]
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1, \text{ for all } 0 \leq u \leq 1. \]
  If we translate all the control points by the same vector \( q \), the position of the new point at the value \( u \) will just be the position of the old value at \( u \), translated by \( q \):
  \[
  Q_k(u) = BF_0(u)(q + p_{k-1}) + BF_1(u)(q + p_k) + BF_2(u)(q + p_{k+1}) + BF_3(u)(q + p_{k+2})
  = \left( BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) \right) q + P_k(u)
  \]
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1, \text{ for all } 0 \leq u \leq 1. \]
  If we translate all the control points by the same vector \( q \), the position of the new point at the value \( u \) will just be the position of the old value at \( u \), translated by \( q \):

  \[
  Q_k(u) = BF_0(u)(q + p_{k-1}) + BF_1(u)(q + p_k) + BF_2(u)(q + p_{k+1}) + BF_3(u)(q + p_{k+2})
  
  = (BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u))q + P_k(u)
  
  = q + P_k(u)
  \]
Comparison: Cardinal vs. Cubic B

Cardinal Splines (t=0)

Cardinal basis functions:

\[
\begin{align*}
BF_0(u) &= -\frac{1}{2} u^3 + u^2 - \frac{1}{2} u \\
BF_1(u) &= \frac{3}{2} u^3 - \frac{5}{2} u^2 + 1 \\
BF_2(u) &= -\frac{3}{2} u^3 + 2u^2 + \frac{1}{2} u \\
BF_3(u) &= \frac{1}{2} u^3 - \frac{1}{2} u^2
\end{align*}
\]

Sum of basis functions:
\[BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1\]

Cubic B-Splines

Cubic B-spline basis functions:

\[
\begin{align*}
BF_0(u) &= -\frac{1}{6} u^3 + \frac{1}{2} u^2 - \frac{1}{2} u + \frac{1}{6} \\
BF_1(u) &= \frac{1}{2} u^3 - u^2 + \frac{2}{3} \\
BF_2(u) &= -\frac{1}{2} u^3 + \frac{1}{2} u^2 + \frac{1}{2} u + \frac{1}{6} \\
BF_3(u) &= \frac{1}{6} u^3
\end{align*}
\]

Sum of basis functions:
\[BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1\]

Partition of unity:
\[P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2}\]
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1, \text{ for all } 0 \leq u \leq 1. \]

• Continuity:
  \[ BF_0(1) = BF_3(0) = 0 \]
  \[ BF_1(1) = BF_0(0) \]
  \[ BF_2(1) = BF_1(0) \]
  \[ BF_3(1) = BF_2(0) \]
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1, \text{ for all } 0 \leq u \leq 1. \]

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  \[ BF_0(1) = BF_3(0) = 0 \]
  \[ BF_1(1) = BF_0(0) \]
  \[ BF_2(1) = BF_1(0) \]
  \[ BF_3(1) = BF_2(0) \]

We need to have the curve \( P_{k+1}(u) \) begin where the curve \( P_k(u) \) ended:

\[ 0 = P_{k+1}(0) - P_k(1) \]
Blending Functions

Properties:

• Translation Commutativity:
  \( BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1, \) for all \( 0 \leq u \leq 1. \)

• Continuity:
  \( BF_0(1) = BF_3(0) = 0 \)
  \( BF_1(1) = BF_0(0) \)
  \( BF_2(1) = BF_1(0) \)
  \( BF_3(1) = BF_2(0) \)

Since this equation has to hold true regardless of the values of \( p_k \), the conditions on the left have to be true.

We need to have the curve \( P_{k+1}(u) \) begin where the curve \( P_k(u) \) ended:

\[
0 = P_{k+1}(0) - P_k(1)
\]

\[
0 = \left( -BF_0(1) \right) p_{k-1} + \left( BF_0(0) - BF_1(1) \right) p_k \\
+ \left( BF_1(0) - BF_2(1) \right) p_{k+1} \\
+ \left( BF_2(0) - BF_3(1) \right) p_{k+2} \\
+ \left( BF_3(0) \right) p_{k+3}
\]
Comparison: Cardinal vs. Cubic B

Cardinal Splines ($t=0$)

- $BF_0(u) = -\frac{1}{2}u^3 + u^2 - \frac{1}{2}u$
- $BF_1(u) = \frac{3}{2}u^3 - \frac{5}{2}u^2 + 1$
- $BF_2(u) = -\frac{3}{2}u^3 + 2u^2 + \frac{1}{2}u$
- $BF_3(u) = \frac{1}{2}u^3 - \frac{1}{2}u^2$

Cardinal B-Splines

- $BF_0(0) = 0$, $BF_0(1) = 0$
- $BF_1(0) = 1$, $BF_1(1) = 0$
- $BF_2(0) = 0$, $BF_2(1) = 1$
- $BF_3(0) = 0$, $BF_3(1) = 0$

Cubic B-Splines

- $BF_0(u) = -\frac{1}{6}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u + \frac{1}{6}$
- $BF_1(u) = \frac{1}{2}u^3 - u^2 + \frac{2}{3}$
- $BF_2(u) = -\frac{1}{2}u^3 + \frac{1}{2}u^2 + \frac{1}{2}u + \frac{1}{6}$
- $BF_3(u) = \frac{1}{6}u^3$

Cubic B-Splines

- $BF_0(0) = \frac{1}{6}$, $BF_0(1) = 0$
- $BF_1(0) = \frac{2}{3}$, $BF_1(1) = \frac{1}{6}$
- $BF_2(0) = \frac{1}{6}$, $BF_2(1) = \frac{2}{3}$
- $BF_3(0) = 0$, $BF_3(1) = \frac{1}{6}$

$P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2}$
Comparison: Cardinal vs. Cubic B

Cardinal Splines (t=0)

\[ BF_0(u) = -\frac{1}{2}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u \]
\[ BF_1(u) = \frac{3}{2}u^3 - \frac{5}{2}u^2 + 1 \]
\[ BF_2(u) = -\frac{3}{2}u^3 + 2u^2 + \frac{1}{2}u \]
\[ BF_3(u) = \frac{1}{2}u^3 - \frac{1}{2}u^2 \]

\[ BF_0'(0) = -\frac{1}{2} \quad BF_0'(1) = 0 \]
\[ BF_1'(0) = 0 \quad BF_1'(1) = -\frac{1}{2} \]
\[ BF_2'(0) = \frac{1}{2} \quad BF_2'(1) = 0 \]
\[ BF_3'(0) = 0 \quad BF_3'(1) = \frac{1}{2} \]

Cubic B-Splines

\[ BF_0(u) = -\frac{1}{6}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u + \frac{1}{6} \]
\[ BF_1(u) = \frac{1}{2}u^3 - \frac{5}{2}u^2 + 1 \]
\[ BF_2(u) = -\frac{3}{2}u^3 + \frac{1}{2}u^2 + \frac{1}{2}u + \frac{1}{6} \]
\[ BF_3(u) = \frac{1}{2}u^3 \]

\[ BF_0'(0) = -\frac{1}{2} \quad BF_0'(1) = 0 \]
\[ BF_1'(0) = 0 \quad BF_1'(1) = -\frac{1}{2} \]
\[ BF_2'(0) = \frac{1}{2} \quad BF_2'(1) = 0 \]
\[ BF_3'(0) = 0 \quad BF_3'(1) = \frac{1}{2} \]

\[ P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2} \]
Comparison: Cardinal vs. Cubic B

Cardinal Splines (t=0)

\[ BF_0(u) = -\frac{1}{2} u^3 + u^2 - \frac{1}{2} u \]
\[ BF_1(u) = \frac{3}{2} u^3 - \frac{5}{2} u^2 + 1 \]
\[ BF_2(u) = -\frac{3}{2} u^3 + 2u^2 + \frac{1}{2} u \]
\[ BF_3(u) = \frac{1}{2} u^3 - \frac{1}{2} u^2 \]

\[ BF_0''(0) = 2 \quad BF_0''(1) = 5 \]
\[ BF_1''(0) = -5 \quad BF_1''(1) = 4 \]
\[ BF_2''(0) = 4 \quad BF_2''(1) = -5 \]
\[ BF_3''(0) = -1 \quad BF_3''(1) = 2 \]

\[ P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2} \]

Cubic B-Splines

\[ BF_0(u) = -\frac{1}{6} u^3 + \frac{1}{2} u^2 - \frac{1}{2} u + \frac{1}{6} \]
\[ BF_1(u) = \frac{1}{2} u^3 - u^2 + \frac{2}{3} \]
\[ BF_2(u) = -\frac{1}{2} u^3 + \frac{1}{2} u^2 + \frac{1}{2} u + \frac{1}{6} \]
\[ BF_3(u) = \frac{1}{6} u^3 \]

\[ BF_0''(0) = 1 \quad BF_0''(1) = 0 \]
\[ BF_1''(0) = -2 \quad BF_1''(1) = 1 \]
\[ BF_2''(0) = 1 \quad BF_2''(1) = -2 \]
\[ BF_3''(0) = 0 \quad BF_3''(1) = 1 \]
Blending Functions

Properties:

• Translation Commutativity:
  \[ BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1, \text{ for all } 0 \leq u \leq 1. \]

• Continuity:
  \[ BF_0(1) = BF_1(0) = BF_2(0) = BF_3(0) = 0 \]
  \[ BF_1(1) = BF_0(0) \]
  \[ BF_2(1) = BF_1(0) \]
  \[ BF_3(1) = BF_2(0) \]
Blending Functions

Properties:

• Translation Commutativity:
  \( BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1, \) for all \( 0 \leq u \leq 1. \)

• Continuity:
  \( BF_0(1) = BF_3(0) = 0 \)
  \( BF_1(1) = BF_0(0) \)
  \( BF_2(1) = BF_1(0) \)
  \( BF_3(1) = BF_2(0) \)

• Convex Hull Containment:
  \( BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0, \) for all \( 0 \leq u \leq 1. \)

This is because a point is inside the convex hull of a collection of points if and only if it can be expressed as the weighted average of the points, where all the weights are non-negative.
Comparison: Cardinal vs. Cubic B

Cardinal Splines (t=0)

Cubic B-Splines

\[ P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2} \]
Comparison: Cardinal vs. Cubic B

Cardinal Splines ($t=0$) | Cubic B-Splines

\[ P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2} \]
Blending Functions

Properties:

• Translation Commutativity:
  - $BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1$, for all $0 \leq u \leq 1$.

• Continuity:
  - $BF_0(1) = BF_3(0) = 0$
  - $BF_1(1) = BF_0(0)$
  - $BF_2(1) = BF_1(0)$
  - $BF_3(1) = BF_2(0)$

• Convex Hull Containment:
  - $BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0$, for all $0 \leq u \leq 1$.

• Interpolation:
  - $BF_0(0) = BF_2(0) = BF_3(0) = 0$
  - $BF_0(1) = BF_1(1) = BF_3(1) = 0$
  - $BF_1(0) = 1$
  - $BF_2(1) = 1$
Blending Functions

Properties:

• Translation Commutativity:
  \( BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1 \), for all \( 0 \leq u \leq 1 \).

• Continuity:
  \( BF_0(1) = BF_3(0) = 0 \)
  \( BF_1(1) = BF_0(0) \)
  \( BF_2(1) = BF_1(0) \)
  \( BF_3(1) = BF_2(0) \)

• Convex Hull Containment:
  \( BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0 \), for all \( 0 \leq u \leq 1 \).

• Interpolation:
  \( BF_0(0) = BF_2(0) = BF_3(0) = 0 \)
  \( BF_0(1) = BF_1(1) = BF_3(1) = 0 \)
  \( BF_1(0) = 1 \)
  \( BF_2(1) = 1 \)

Because we want the spline segments to satisfy:

\( P_k(0) = p_{k+1} \)
\( P_k(1) = p_{k+2} \)
Comparison: Cardinal vs. Cubic B

Cardinal Splines (t=0)

- \( BF_0(u) = -\frac{1}{2} u^3 + \frac{1}{2} u^2 - \frac{1}{2} u \)
- \( BF_1(u) = \frac{3}{2} u^3 - \frac{5}{2} u^2 + 1 \)
- \( BF_2(u) = -\frac{3}{2} u^3 + 2u^2 + \frac{1}{2} u \)
- \( BF_3(u) = \frac{1}{2} u^3 - \frac{1}{2} u^2 \)

Cubic B-Splines

- \( BF_0(u) = -\frac{1}{6} u^3 + \frac{1}{2} u^2 - \frac{1}{2} u + \frac{1}{6} \)
- \( BF_1(u) = \frac{1}{2} u^3 - u^2 + 2 \)
- \( BF_2(u) = -\frac{1}{2} u^3 + \frac{1}{2} u^2 + 2u + \frac{1}{6} \)
- \( BF_3(u) = \frac{1}{6} u^3 \)

Boundary conditions:

- \( BF_0(0) = 0 \)
- \( BF_0(1) = 0 \)
- \( BF_1(0) = 1 \)
- \( BF_1(1) = 0 \)
- \( BF_2(0) = 0 \)
- \( BF_2(1) = 1 \)
- \( BF_3(0) = 0 \)
- \( BF_3(1) = 0 \)

Piecewise polynomial function:

\[ P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2} \]
Summary

• A spline is a *piecewise polynomial function* whose derivatives satisfy some *continuity constraints* across curve junctions.

• Looked at specification for 3 splines:
  - Hermite
  - Cardinal
  - Uniform Cubic B-Spline

\[ \begin{align*}
\text{Hermite} & \quad \text{Interpolating, cubic, } C^1 \\
\text{Cardinal} & \quad \text{Approximating, convex-hull containment, cubic, } C^2
\end{align*} \]