Barycentric Coordinates (and Some Texture Mapping)

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CS 4810: Graphics

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Triangles

These are the basic building blocks of 3D models.

- Often 3D models are complex, and the surfaces are represented by a triangulated approximation.
# Triangles

A triangle is defined by three non-collinear vertices:

- Any point $q$ in the triangle is on the line segment between one vertex and some other point $q'$ on the opposite edge.
Barycentric Coordinates

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- Any point on the triangle can be expressed as:
  - $q = \{ \alpha p_1 + \beta p_2 + \gamma p_3 \mid \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \geq 0 \}$
Barycentric Coordinates

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\[
\alpha p_1 + \beta p_2 + \gamma p_3 = \alpha p_1 + (1 - \alpha) \left( \frac{\beta p_2 + \gamma p_3}{1 - \alpha} \right)
\]
Barycentric Coordinates

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A point \( q \) on the segment between \( p_1 \) and \( q' \)
Barycentric Coordinates

The barycentric coordinates of a point \( q \):

\[
q = \alpha p_1 + \beta p_2 + \gamma p_3
\]

allow us to express \( q \) as a weighted average of the vertices of the triangles.
Barycentric Coordinates

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Questions:

• What happens if \( \alpha, \beta, \) or \( \gamma < 0 \)?
Barycentric Coordinates

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Questions:

• What happens if \( \alpha, \beta, \) or \( \gamma < 0? \)
  
  \( q \) is not inside the triangle but it is in the plane spanned by \( p_1, p_2, \) and \( p_3. \)
Barycentric Coordinates

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Questions:

• What happens if \( \alpha, \beta, \) or \( \gamma < 0 \)?
• What happens if \( \alpha + \beta + \gamma \neq 1 \)?
Barycentric Coordinates

Any point on the triangle can be expressed as:

\[ q = \{ \alpha \mathbf{p}_1 + \beta \mathbf{p}_2 + \gamma \mathbf{p}_3 \mid \alpha + \beta + \gamma = 1, \, \alpha, \beta, \gamma \geq 0 \} \]

Questions:

• What happens if \( \alpha, \beta, \) or \( \gamma < 0 \)?

• What happens if \( \alpha + \beta + \gamma \neq 1 \)?

  \( q \) is not in the plane spanned by \( p_1, p_2, \) and \( p_3 \).
Barycentric Coordinates

Any point on the triangle can be expressed as:

\[ q = \{ \alpha p_1 + \beta p_2 + \gamma p_3 \mid \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \geq 0 \} \]

Questions:

• What happens if \( \alpha, \beta, \) or \( \gamma < 0 \)?

• What happens if \( \alpha + \beta + \gamma \neq 1 \)?

**Note:** If we force \( \alpha = 1 - \beta - \gamma \), we always get \( \alpha + \beta + \gamma = 1 \) so the point \( q \) is always in the plane containing the triangle.
Barycentric Coordinates

Barycentric coordinates are needed in:

- Ray-Tracing, to test for intersection
- Rendering, to interpolate triangle information
Barycentric Coordinates

Barycentric coordinates are needed in:

• Ray-Tracing, to test for intersection

• Rendering, to interpolate triangle information

Float TriangleIntersect(Ray r, Triangle tgl) {
    Plane p = PlaneContaining(tgl);
    Float t = IntersectionDistance(r, p);
    if (t < 0) { return -1; }
    else {
        (α, β, γ) = Barycentric(r(t), tgl);
        if (α < 0 or β < 0 or γ < 0) { return -1; }
        else { return t; }
    }
}
Barycentric Coordinates

Barycentric coordinates are needed in:

• Ray-Tracing, to test for intersection

• Rendering, to interpolate triangle information
  • In 3D models, information is often associated with vertices rather than triangles (e.g. color, normals, etc.)
Barycentric Coordinates

For example:

• We could associate the same normal/color to every point on the face of a triangle by computing:

\[
\mathbf{n} = \frac{(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1)}{\| (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) \|}
\]
Barycentric Coordinates

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\mathbf{n} = \frac{\mathbf{(p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1)}{\left\| (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) \right\|}
\]

This gives rise to flat shading/coloring across the faces.
Barycentric Coordinates

Instead:

- We could associate normals to every vertex:

\[ T = ((p_1, n_1), (p_2, n_2), (p_3, n_3)) \]

so that the normal at some point \( q \) in the triangle is the interpolation of the normals at the vertices:

\[
n(q) = \frac{\alpha(q)n_1 + \beta(q)n_2 + \gamma(q)n_3}{\|\alpha(q)n_1 + \beta(q)n_2 + \gamma(q)n_3\|}
\]
Barycentric Coordinates

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  \[ T = \left( (p_1, n_1), (p_2, n_2), (p_3, n_3) \right) \]
  so that the normal at some point \( q \) in the triangle is the interpolation of the normals at the vertices:
Barycentric Coordinates

So given the points $p_1$, $p_2$, and $p_3$, how do we compute the barycentric coordinates of a point $q$ in the plane spanned by $p_1$, $p_2$, and $p_3$?

Matrix Inversion:

We can approach this as a linear system with three equations and two unknowns:

\[ q_x = (1 - \beta - \gamma) p_{1x} + \beta p_{2x} + \gamma p_{2x} \]
\[ q_y = (1 - \beta - \gamma) p_{1y} + \beta p_{2y} + \gamma p_{2y} \]
\[ q_z = (1 - \beta - \gamma) p_{1z} + \beta p_{2z} + \gamma p_{2z} \]
Barycentric Coordinates

So given the points $p_1$, $p_2$, and $p_3$, how do we compute the barycentric coordinates of a point $q$ in the plane spanned by $p_1$, $p_2$, and $p_3$?

(Signed) Area Ratios:

$$\alpha = \frac{A_1}{A_1 + A_2 + A_3}$$

$$\beta = \frac{A_2}{A_1 + A_2 + A_3}$$

$$\gamma = \frac{A_3}{A_1 + A_2 + A_3}$$
Barycentric Coordinates

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Solving this equation requires computing the areas of three triangles for every point $q$. (DERIVATION IN CLASS)
Texture Mapping (Briefly, More Later)
Textures

- How can we go about drawing surfaces with complex detail?
Textures

• How can we go about drawing surfaces with complex detail?

• We could tessellate the sphere in a complex fashion and then associate the appropriate material properties to each vertex.

Target Model

Complex Surface
Textures

• How can we go about drawing surfaces with complex detail?

• We could use a simple tessellation and use the location of surface points to look up the appropriate color values.
Textures

• Advantages:
  - The 3D model remains simple
  - It is easier to design/modify a texture image than it is to design/modify a surface in 3D.
Another Example: Brick Wall
Another Example: Brick Wall
2D Texture

• Coordinates described by variables $s$ and $t$ and range over interval $(0,1)$

• Texture elements are called *texels*

• Often 4 bytes (rgba) per texel
Texture Mapping a Sphere

• How do you generate texture coordinates at each intersection point?