Barycentric Coordinates
(and Some Texture Mapping)

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CS 4810: Graphics

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Triangles

These are the basic building blocks of 3D models.

- Often 3D models are complex, and the surfaces are represented by a triangulated approximation.
Triangles

A triangle is defined by three non-collinear vertices:

- Any point $q$ in the triangle is on the line segment between one vertex and some other point $q'$ on the opposite edge.
Barycentric Coordinates

A triangle is defined by three non-collinear vertices:

• Any point $q$ in the triangle is on the line segment between one vertex and some other point $q'$ on the opposite edge.

• Any point on the triangle can be expressed as:
  • $q = \{ \alpha p_1 + \beta p_2 + \gamma p_3 \mid \alpha + \beta + \gamma = 1, \ \alpha, \beta, \gamma \geq 0 \}$
Barycentric Coordinates

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\[
\alpha p_1 + \beta p_2 + \gamma p_3 = \alpha p_1 + \left(1 - \alpha\right)\left(\frac{\beta p_2 + \gamma p_3}{1 - \alpha}\right)
\]
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$$\alpha p_1 + \beta p_2 + \gamma p_3 = \alpha p_1 + (1-\alpha) \left( \frac{\beta p_2 + \gamma p_3}{1-\alpha} \right)$$

A point $q'$ on the segment between $p_2$ and $p_3$
Barycentric Coordinates

A triangle is defined by three non-collinear vertices:

- Any point \( q \) in the triangle is on the line segment between one vertex and some other point \( q' \) on the opposite edge.

- Any point on the triangle can be expressed as:
  - \( q = \{ \alpha p_1 + \beta p_2 + \gamma p_3 \mid \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \geq 0 \} \)

\[
\alpha p_1 + \beta p_2 + \gamma p_3 = \alpha p_1 + (1 - \alpha) \left( \frac{\beta p_2 + \gamma p_3}{\beta + \gamma} \right)
\]

A point \( q \) on the segment between \( p_1 \) and \( q' \)
Barycentric Coordinates

The barycentric coordinates of a point \( q \):

\[
q = \alpha p_1 + \beta p_2 + \gamma p_3
\]

allow us to express \( q \) as a weighted average of the vertices of the triangles.
Barycentric Coordinates

Any point on the triangle can be expressed as:
• \( q = \{ \alpha p_1 + \beta p_2 + \gamma p_3 \mid \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \geq 0 \} \)

Questions:
• What happens if \( \alpha, \beta, \) or \( \gamma < 0 \)?
Barycentric Coordinates

Any point on the triangle can be expressed as:

\[ q = \alpha p_1 + \beta p_2 + \gamma p_3 \mid \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \geq 0 \]

Questions:

• What happens if \( \alpha, \beta, \) or \( \gamma < 0 \)?
  
  \( oq \) is not inside the triangle but it is in the plane spanned by \( p_1, p_2, \) and \( p_3 \).
Barycentric Coordinates

Any point on the triangle can be expressed as:

\[ q = \{ \alpha p_1 + \beta p_2 + \gamma p_3 | \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \geq 0 \} \]

Questions:

• What happens if \( \alpha, \beta, \) or \( \gamma < 0 \)?
• What happens if \( \alpha + \beta + \gamma \neq 1 \)?
Barycentric Coordinates

Any point on the triangle can be expressed as:

\[ q = \alpha p_1 + \beta p_2 + \gamma p_3 \mid \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \geq 0 \]

Questions:

• What happens if \( \alpha, \beta, \) or \( \gamma < 0 \)?

• What happens if \( \alpha + \beta + \gamma \neq 1 \)?
  
  \( q \) is not in the plane spanned by \( p_1, p_2, \) and \( p_3 \).
Barycentric Coordinates

Any point on the triangle can be expressed as:

\[ q = \{ \alpha p_1 + \beta p_2 + \gamma p_3 \mid \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \geq 0 \} \]

Questions:

• What happens if \( \alpha, \beta, \) or \( \gamma < 0 \)?

• What happens if \( \alpha + \beta + \gamma \neq 1 \)?

Note: If we force \( \alpha = 1 - \beta - \gamma \), we always get \( \alpha + \beta + \gamma = 1 \) so the point \( q \) is always in the plane containing the triangle.
Barycentric Coordinates

Barycentric coordinates are needed in:

• Ray-Tracing, to test for intersection
• Rendering, to interpolate triangle information
Barycentric Coordinates

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• Ray-Tracing, to test for intersection

• Rendering, to interpolate triangle information

```c
Float TriangleIntersect(Ray r, Triangle tgl) {
    Plane p = PlaneContaining(tgl);
    Float t = IntersectionDistance(r, p);
    if (t < 0) { return -1; }
    else {
        (α, β, γ) = Barycentric(r(t), tgl);
        if (α < 0 || β < 0 || γ < 0) { return -1; }
        else { return t; }
    }
}
```
Barycentric Coordinates

Barycentric coordinates are needed in:

• Ray-Tracing, to test for intersection

• Rendering, to interpolate triangle information

  In 3D models, information is often associated with vertices rather than triangles (e.g. color, normals, etc.)
Barycentric Coordinates

For example:

- We could associate the same normal/color to every point on the face of a triangle by computing:

\[
n = \frac{(p_2 - p_1) \times (p_3 - p_1)}{\| (p_2 - p_1) \times (p_3 - p_1) \|}
\]
Barycentric Coordinates

For example:

• We could associate the same normal/color to every point on the face of a triangle by computing:

\[ n = \frac{(p_2 - p_1) \times (p_3 - p_1)}{||(p_2 - p_1) \times (p_3 - p_1)||} \]

This gives rise to flat shading/coloring across the faces.
Barycentric Coordinates

Instead:

- We could associate normals to every vertex:
  \[ T = \{(p_1, n_1), (p_2, n_2), (p_3, n_3)\} \]
  so that the normal at some point \( q \) in the triangle is the interpolation of the normals at the vertices:

  \[
  n(q) = \frac{\alpha(q)n_1 + \beta(q)n_2 + \gamma(q)n_3}{\|\alpha(q)n_1 + \beta(q)n_2 + \gamma(q)n_3\|}
  \]
Barycentric Coordinates

Instead:

• We could associate normals to every vertex:

\[ T = ((p_1, n_1), (p_2, n_2), (p_3, n_3)) \]

so that the normal at some point \( q \) in the triangle is the interpolation of the normals at the vertices:

Triangle Normals

Interpolated Point Normals
Barycentric Coordinates

So given the points $p_1$, $p_2$, and $p_3$, how do we compute the barycentric coordinates of a point $q$ in the plane spanned by $p_1$, $p_2$, and $p_3$?

Matrix Inversion:

We can approach this as a linear system with three equations and two unknowns:

$$q_x = (1 - \beta - \gamma) p_{1x} + \beta p_{2x} + \gamma p_{2x}$$

$$q_y = (1 - \beta - \gamma) p_{1y} + \beta p_{2y} + \gamma p_{2y}$$

$$q_z = (1 - \beta - \gamma) p_{1z} + \beta p_{2z} + \gamma p_{2z}$$
Barycentric Coordinates

So given the points \( p_1, p_2, \) and \( p_3 \), how do we compute the barycentric coordinates of a point \( q \) in the plane spanned by \( p_1, p_2, \) and \( p_3 \)?

\[(\text{Signed) Area Ratios:}\]

\[\alpha = \frac{A_1}{A_1 + A_2 + A_3}\]

\[\beta = \frac{A_2}{A_1 + A_2 + A_3}\]

\[\gamma = \frac{A_3}{A_1 + A_2 + A_3}\]
Barycentric Coordinates

So given the points \( p_1, p_2, \) and \( p_3, \) how do we compute the barycentric coordinates of a point \( q \) in the plane spanned by \( p_1, p_2, \) and \( p_3? \)

(Signed) Area Ratios:

\[
\alpha = \frac{A_1}{A_1 + A_2 + A_3}
\]
\[
\beta = \frac{A_2}{A_1 + A_2 + A_3}
\]
\[
\gamma = \frac{A_3}{A_1 + A_2 + A_3}
\]

Solving this equation requires computing the areas of three triangles for every point \( q. \) (DERIVATION IN CLASS)
Texture Mapping (Briefly, More Later)
Textures

- How can we go about drawing surfaces with complex detail?
Textures

• How can we go about drawing surfaces with complex detail?

• We could tessellate the sphere in a complex fashion and then associate the appropriate material properties to each vertex
Textures

• How can we go about drawing surfaces with complex detail?

• We could use a simple tessellation and use the location of surface points to look up the appropriate color values
Textures

• Advantages:
  o The 3D model remains simple
  o It is easier to design/modify a texture image than it is to design/modify a surface in 3D.
Another Example: Brick Wall
Another Example: Brick Wall
2D Texture

- Coordinates described by variables $s$ and $t$ and range over interval $(0,1)$
- Texture elements are called *texels*
- Often 4 bytes (rgba) per texel
Texture Mapping a Sphere

- How do you generate texture coordinates at each intersection point?