

**Derivation of the 2-D Model of Walker Dynamics with 3-D
Forces and Moments Measured at the Handles**

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INTRODUCTION

The walker, shown in Figure 1, consists of the frame, the power train and electronic box that are connected to the frame tightly, so we can consider them as a rigid body. The two rear wheels are connected to the frame by revolute joints. Considering the low mass and low speed of the walker usage, the two rear wheels are not considered as additional rigid bodies in this model, but their inertial property are included with the main rigid body (rigid body 1). The front caster wheel is modeled as the second rigid body to reflect the offset effect of the caster wheel to the user. So in this model, there are two rigid bodies involved. Two local coordinate systems and one global coordinate system will be defined to illustration their kinematic relations (figure 2). The three wheels may also have nonholonomic constraints on the system, depending upon the friction forces involved.



Figure 1. Photograph of the Walker Used For Model Development

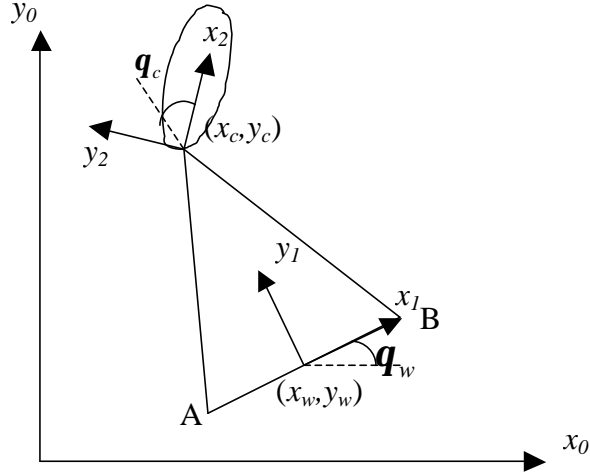


Figure 2 orientation of the local and global coordinate systems used in the model

Derivation of homogenous part of the system dynamics

Consider rigid body 1 representing the majority parts of the walker, the transformation from the global coordinate system $[x_0, y_0]^T$ to local coordinate system $[x_1, y_1]^T$ is given by:

$$\begin{Bmatrix} 1 \\ x_0 \\ y_0 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{Bmatrix} 1 \\ x_1 \\ y_1 \end{Bmatrix}, \quad (1)$$

Where: $[x_w, y_w]^T$ is the position of the origin of the local coordinate system defined in rigid body 1 relative to the global coordinate system (figure 2).

So the transformation matrix is: $A_{01} = \begin{bmatrix} 1 & 0 & 0 \\ x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix}_{3 \times 3}$ (2)

For the rigid body 2 representing caster wheel, the transformation from the global coordinate system $[x_0, y_0]^T$ to local coordinate system $[x_2, y_2]^T$ is given by:

$$\begin{aligned}
\begin{Bmatrix} 1 \\ x_0 \\ y_0 \end{Bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{Bmatrix} 1 \\ x_1 \\ y_1 \end{Bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{bmatrix} 1 \\ x_c \cos\left(\frac{\mathbf{p}}{2} - \mathbf{q}_c\right) - \sin\left(\frac{\mathbf{p}}{2} - \mathbf{q}_c\right) \\ y_c \sin\left(\frac{\mathbf{p}}{2} - \mathbf{q}_c\right) \cos\left(\frac{\mathbf{p}}{2} - \mathbf{q}_c\right) \end{bmatrix} \begin{Bmatrix} 1 \\ x_2 \\ y_2 \end{Bmatrix} \quad (3) \\
&= A_{01} A_{12} \begin{Bmatrix} 1 \\ x_2 \\ y_2 \end{Bmatrix}
\end{aligned}$$

where: $A_{12} = \begin{bmatrix} 1 & 0 & 0 \\ x_c & \sin \mathbf{q}_c & -\cos \mathbf{q}_c \\ y_c & \cos \mathbf{q}_c & \sin \mathbf{q}_c \end{bmatrix}$ are the transformation matrix from local coordinate system 2 to local coordinate system 1.

In 2D case, the kinematic energy of the rigid body can be expressed as^[2]:

$$T = \frac{1}{2} \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\mathbf{q}} \end{Bmatrix}^T \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}_{3 \times 3} \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\mathbf{q}} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\mathbf{q}} \end{Bmatrix}^T M \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\mathbf{q}} \end{Bmatrix} \quad (4)$$

where:

$$M = \begin{bmatrix} m & 0 & -m(u \sin \mathbf{q} + v \cos \mathbf{q}) \\ 0 & m & m(u \cos \mathbf{q} - v \sin \mathbf{q}) \\ -m(u \sin \mathbf{q} + v \cos \mathbf{q}) & m(u \cos \mathbf{q} - v \sin \mathbf{q}) & I_z \end{bmatrix} \quad (5)$$

m is the mass of the rigid body. $[u, v]$ is the position of the center of body mass in the body local coordinate system. I_z is principle inertial property of the body around axis of rotation. (x, y) are the coordinates of the origin of the local coordinate system in the global system. \mathbf{q} is the absolute rotation angle of the local coordinate system in the global system.

We define the origin of the walker frame to be in the middle point $\{x_w \ y_w\}^T$ of the rotation axis of the two back wheels, then we choose position $\{x_w \ y_w\}^T$ and rotation angle \mathbf{q}_w of the origin of the frame, together with the rotation angle \mathbf{q}_c of the caster wheel relative to walker frame to be the 4 general coordinates needed in this dynamic model.

For the rigid body 1, three general coordinates $\{x_w \ y_w \ \mathbf{q}_w\}^T$ are related with the kinematic energy of rigid body 1.

$$T_1 = \frac{1}{2} \begin{Bmatrix} \dot{x}_w \\ \dot{y}_w \\ \dot{\mathbf{q}}_w \end{Bmatrix}^T M_1 \begin{Bmatrix} \dot{x}_w \\ \dot{y}_w \\ \dot{\mathbf{q}}_w \end{Bmatrix} = \frac{1}{2} \dot{\bar{x}}^T M_1 \dot{\bar{x}} \quad (6)$$

where: $\bar{x} = \begin{Bmatrix} x_w \\ y_w \\ \mathbf{q}_w \end{Bmatrix}$,

$$M_1 = \begin{bmatrix} m_w & 0 & -m_w(u_w \sin \mathbf{q}_w + v_w \cos \mathbf{q}_w) \\ 0 & m_w & m_w(u_w \cos \mathbf{q}_w - v_w \sin \mathbf{q}_w) \\ -m_w(u_w \sin \mathbf{q}_w + v_w \cos \mathbf{q}_w) & m_w(u_w \cos \mathbf{q}_w - v_w \sin \mathbf{q}_w) & I_1 \end{bmatrix}$$

Because M_1 is a positive definite and symmetric matrix, we can use the following property of this kind of matrix:

$$\left\{ \frac{\partial T_1}{\partial \dot{x}_i} \right\} = \frac{1}{2} \left(2 \left[\frac{\partial \dot{\bar{x}}^T}{\partial \dot{x}_i} \right] [M_1] \left[\frac{\dot{\bar{x}}}{\dot{x}_i} \right] + \left[\frac{\dot{\bar{x}}^T}{\dot{x}_i} \right] \left[\frac{\partial M_1}{\partial \dot{x}_i} \right] \left[\frac{\dot{\bar{x}}}{\dot{x}_i} \right] \right) = \text{the } i\text{th element of } [M_1] \left[\frac{\dot{\bar{x}}}{\dot{x}_i} \right] \quad (7)$$

$$\left\{ \frac{\partial T_1}{\partial x_i} \right\} = \frac{1}{2} \left(2 \left[\frac{\partial \dot{\bar{x}}^T}{\partial x_i} \right] [M_1] \left[\frac{\dot{\bar{x}}}{\dot{x}_i} \right] + \left[\frac{\dot{\bar{x}}^T}{\dot{x}_i} \right] \left[\frac{\partial M_1}{\partial x_i} \right] \left[\frac{\dot{\bar{x}}}{\dot{x}_i} \right] \right) = \frac{1}{2} \left[\frac{\dot{\bar{x}}^T}{\dot{x}_i} \right] \left[\frac{\partial M_1}{\partial x_i} \right] \left[\frac{\dot{\bar{x}}}{\dot{x}_i} \right]$$

So

$$\frac{d}{dt} \left(\frac{\partial T_1}{\partial \dot{\bar{x}}} \right) = \frac{d}{dt} ([M_1] \left[\frac{\dot{\bar{x}}}{\dot{x}_i} \right]) = [M_1] \left[\frac{\ddot{\bar{x}}}{\ddot{x}_i} \right] + [\dot{M}_1] \left[\frac{\dot{\bar{x}}}{\dot{x}_i} \right] \quad (8)$$

where: $[\dot{M}_1] = \sum_{i=1}^3 \frac{\partial M_1}{\partial x_i} \dot{x}_i$

$$\left\{ \frac{\partial T_1}{\partial \bar{x}} \right\} = \frac{1}{2} \left\{ \begin{array}{l} \left[\frac{\dot{\bar{x}}^T}{\dot{x}_1} \right] \left[\frac{\partial M_1}{\partial x_1} \right] \left[\frac{\dot{\bar{x}}}{\dot{x}_1} \right] \\ \left[\frac{\dot{\bar{x}}^T}{\dot{x}_2} \right] \left[\frac{\partial M_1}{\partial x_2} \right] \left[\frac{\dot{\bar{x}}}{\dot{x}_2} \right] \\ \left[\frac{\dot{\bar{x}}^T}{\dot{x}_3} \right] \left[\frac{\partial M_1}{\partial x_3} \right] \left[\frac{\dot{\bar{x}}}{\dot{x}_3} \right] \end{array} \right\} \quad (9)$$

For rigid body 2, all the general coordinates are related with its kinetic energy. Following shows how to represent the kinetic energy with these general coordinates.

$$\begin{cases} 1 \\ x_{co} \\ y_{co} \end{cases} = A_{01} \begin{cases} 1 \\ x_c \\ y_c \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{cases} 1 \\ x_c \\ y_c \end{cases}$$

$$\text{So: } \begin{cases} 0 \\ \dot{x}_{co} \\ \dot{y}_{co} \end{cases} = \dot{A}_{01} \begin{cases} 1 \\ x_c \\ y_c \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ \dot{x}_w & -\sin \mathbf{q}_w \dot{\mathbf{q}}_w & -\cos \mathbf{q}_w \dot{\mathbf{q}}_w \\ \dot{y}_w & \cos \mathbf{q}_w \dot{\mathbf{q}}_w & -\sin \mathbf{q}_w \dot{\mathbf{q}}_w \end{bmatrix} \begin{cases} 1 \\ x_c \\ y_c \end{cases}$$

Thus the first derivatives of the position and angle of the origin of the caster wheel can be represented as:

$$\begin{cases} \dot{x}_{co} \\ \dot{y}_{co} \\ \dot{\mathbf{q}}_{co} \end{cases} = \begin{bmatrix} 1 & 0 & -x_c \sin \mathbf{q}_w - y_c \cos \mathbf{q}_w & 0 \\ 0 & 1 & x_c \cos \mathbf{q}_w - y_c \sin \mathbf{q}_w & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{cases} \dot{x}_w \\ \dot{y}_w \\ \dot{\mathbf{q}}_w \\ \dot{\mathbf{q}}_c \end{cases} \quad (10)$$

$$= \mathbf{r} \begin{cases} \dot{x}_w \\ \dot{y}_w \\ \dot{\mathbf{q}}_w \\ \dot{\mathbf{q}}_c \end{cases}$$

Then the kinetic energy of rigid body 2 can be written as:

$$T_2 = \frac{1}{2} \begin{cases} \dot{x}_{co} \\ \dot{y}_{co} \\ \dot{\mathbf{q}}_{co} \end{cases}^T M \begin{cases} \dot{x}_{co} \\ \dot{y}_{co} \\ \dot{\mathbf{q}}_{co} \end{cases} = \frac{1}{2} \begin{cases} \dot{x}_w \\ \dot{y}_w \\ \dot{\mathbf{q}}_w \\ \dot{\mathbf{q}}_c \end{cases}^T \mathbf{r}^T M \mathbf{r} \begin{cases} \dot{x}_w \\ \dot{y}_w \\ \dot{\mathbf{q}}_w \\ \dot{\mathbf{q}}_c \end{cases} = \frac{1}{2} \begin{cases} \dot{x}_w \\ \dot{y}_w \\ \dot{\mathbf{q}}_w \\ \dot{\mathbf{q}}_c \end{cases}^T M_2 \begin{cases} \dot{x}_w \\ \dot{y}_w \\ \dot{\mathbf{q}}_w \\ \dot{\mathbf{q}}_c \end{cases} = \frac{1}{2} \dot{\bar{x}}^T M_2 \dot{\bar{x}} \quad (11)$$

$$\text{where: } \bar{x} = \begin{cases} x_w \\ y_w \\ \mathbf{q}_w \\ \mathbf{q}_c \end{cases}, \quad M_2 = \mathbf{r}^T M \mathbf{r}$$

Because M_2 is a positive definite and symmetric matrix, we can use the following property of this kind of matrix:

$$\left\{ \frac{\partial T_2}{\partial \dot{x}_i} \right\} = \frac{1}{2} \left(2 \left[\frac{\partial \dot{\bar{x}}^T}{\partial \dot{x}_i} \right] \{ [M_2] \dot{\bar{x}} + \dot{\bar{x}}^T \} \left[\frac{\partial M_2}{\partial \dot{x}_i} \right] \dot{\bar{x}} \right) = \text{the } i\text{th element of } [M_2] \dot{\bar{x}} \quad (12)$$

$$\left\{ \frac{\partial T_2}{\partial x_i} \right\} = \frac{1}{2} \left(2 \left[\frac{\partial \dot{\bar{x}}^T}{\partial x_i} \right] \{ [M_2] \dot{\bar{x}} + \dot{\bar{x}}^T \} \left[\frac{\partial M_2}{\partial x_i} \right] \dot{\bar{x}} \right) = \frac{1}{2} \dot{\bar{x}}^T \left[\frac{\partial M_2}{\partial x_i} \right] \dot{\bar{x}}$$

So

$$\frac{d}{dt} \left(\frac{\partial T_2}{\partial \dot{\bar{x}}} \right) = \frac{d}{dt} ([M_2] \dot{\bar{x}}) = [\dot{M}_2] \dot{\bar{x}} + [M_2] \ddot{\bar{x}} \quad (13)$$

where: $[\dot{M}_2] = \sum_{i=1}^4 \frac{\partial M_2}{\partial x_i} \dot{x}_i$

$$\left\{ \frac{\partial T_2}{\partial \dot{\bar{x}}} \right\} = \frac{1}{2} \begin{Bmatrix} \left\{ \dot{\bar{x}}^T \right\} \left[\frac{\partial M_2}{\partial x_1} \right] \left\{ \dot{\bar{x}} \right\} \\ \left\{ \dot{\bar{x}}^T \right\} \left[\frac{\partial M_2}{\partial x_2} \right] \left\{ \dot{\bar{x}} \right\} \\ \left\{ \dot{\bar{x}}^T \right\} \left[\frac{\partial M_2}{\partial x_3} \right] \left\{ \dot{\bar{x}} \right\} \\ \left\{ \dot{\bar{x}}^T \right\} \left[\frac{\partial M_2}{\partial x_4} \right] \left\{ \dot{\bar{x}} \right\} \end{Bmatrix} \quad (14)$$

Put the above equations into LaGrange equations, and synthesize them together, we have the following governing differential equations for this system:

$$[M_s] \{\ddot{\bar{x}}\} + C(\bar{x}, \dot{\bar{x}}) = \{0\} \quad (15)$$

where: $M_s = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix}_{4 \times 4}$ is a new mass matrix of the system which has

positive-definite and symmetric characteristics.

$$\begin{aligned} M_{11} &= m_1 + m_2 \\ M_{12} &= 0 \\ M_{13} &= -m_2 [x_c \sin \mathbf{q}_w + y_c \cos \mathbf{q}_w + u_2 \cos(\mathbf{q}_c - \mathbf{q}_w) + v_2 \sin(\mathbf{q}_c - \mathbf{q}_w)] - m_1 (u_1 \sin \mathbf{q}_w + v_1 \cos \mathbf{q}_w) \\ M_{14} &= m_2 [u_2 \cos(\mathbf{q}_c - \mathbf{q}_w) + v_2 \sin(\mathbf{q}_c - \mathbf{q}_w)] \\ M_{22} &= m_1 + m_2 \\ M_{23} &= m_2 [x_c \cos \mathbf{q}_w - y_c \sin \mathbf{q}_w + u_2 \sin(\mathbf{q}_c - \mathbf{q}_w) - v_2 \cos(\mathbf{q}_c - \mathbf{q}_w)] + m_1 (u_1 \cos \mathbf{q}_w - v_1 \sin \mathbf{q}_w) \\ M_{24} &= -m_2 [u_2 \sin(\mathbf{q}_c - \mathbf{q}_w) - v_2 \cos(\mathbf{q}_c - \mathbf{q}_w)] \\ M_{33} &= I_1 + I_2 + m_2 (x_c^2 + y_c^2) + 2m_2 [(u_2 x_c + v_2 y_c) \sin \mathbf{q}_c + (u_2 y_c - v_2 x_c) \cos \mathbf{q}_c] \\ M_{34} &= -m_2 \left\{ \begin{aligned} & (x_c \sin \mathbf{q}_w + y_c \cos \mathbf{q}_w) [u_2 \cos(\mathbf{q}_c - \mathbf{q}_w) + v_2 \sin(\mathbf{q}_c - \mathbf{q}_w)] \\ & + (x_c \cos \mathbf{q}_w - y_c \sin \mathbf{q}_w) [u_2 \sin(\mathbf{q}_c - \mathbf{q}_w) - v_2 \cos(\mathbf{q}_c - \mathbf{q}_w)] \end{aligned} \right\} - I_2 \\ M_{44} &= I_2 \end{aligned} \quad (16)$$

C is the coupling items:

$$C = \left\{ \begin{array}{l} -m_2 \left\{ (x_c \cos \mathbf{q}_w - y_c \sin \mathbf{q}_w) \dot{\mathbf{q}}_w^2 + [u_2 \sin(\mathbf{q}_c - \mathbf{q}_w) - v_2 \cos(\mathbf{q}_c - \mathbf{q}_w)] (\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w)^2 \right\} \\ -m_1 (u_1 \cos \mathbf{q}_w - v_1 \sin \mathbf{q}_w) \dot{\mathbf{q}}_w^2; \\ -m_2 \left\{ (x_c \sin \mathbf{q}_w + y_c \cos \mathbf{q}_w) \dot{\mathbf{q}}_w^2 + [u_2 \cos(\mathbf{q}_c - \mathbf{q}_w) + v_2 \sin(\mathbf{q}_c - \mathbf{q}_w)] (\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w)^2 \right\} \\ -m_1 (u_1 \sin \mathbf{q}_w + v_1 \cos \mathbf{q}_w) \dot{\mathbf{q}}_w^2; \\ m_2 [(x_c v_2 - y_c u_2) \sin \mathbf{q}_c + (x_c u_2 + y_c v_2) \cos \mathbf{q}_c] (2\dot{\mathbf{q}}_w - \dot{\mathbf{q}}_c) \dot{\mathbf{q}}_c; \\ -m_2 [(x_c u_2 + y_c v_2) \cos \mathbf{q}_c + (x_c v_2 - y_c u_2) \sin \mathbf{q}_c] \dot{\mathbf{q}}_w^2 \end{array} \right\}_{4 \times 1} \quad (17)$$

Nonholonomic constraints in the system

In this model, there exist 3 *possible* nonholonomic constraints in the system, two for the back wheels and one for the front caster wheel.

A. First, the nonholonomic constraints for the two back wheels can be expressed as:

$$\begin{cases} \vec{V}_A \cdot \vec{r}_{AB} = 0 \\ \vec{V}_B \cdot \vec{r}_{BA} = 0 \end{cases} \quad (18)$$

where point A and point B at the left and right point where rear wheel is mounted, respectively.

For point A in rigid body 1:

$$\begin{cases} 1 \\ x_{A0} \\ y_{A0} \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{cases} 1 \\ -\frac{L}{2} \\ 0 \end{cases}, \quad (19)$$

For point B in rigid body 1:

$$\begin{cases} 1 \\ x_{B0} \\ y_{B0} \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{cases} 1 \\ \frac{L}{2} \\ 0 \end{cases}, \quad (20)$$

Then \vec{r}_{AB} can be expressed in the following way: (19)-(20)

$$\begin{cases} 0 \\ x_{A0} - x_{B0} \\ y_{A0} - y_{B0} \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{cases} 0 \\ -L \\ 0 \end{cases} \quad (21)$$

Then velocity \vec{V}_A, \vec{V}_B can be expressed in the following way:

$$\begin{cases} \dot{x}_{A0} \\ \dot{y}_{A0} \end{cases} = [\mathcal{Q}_x \dot{x}_w + \mathcal{Q}_y \dot{y}_w + \mathcal{Q}_q \dot{\mathbf{q}}_w] A_{01} \begin{cases} 1 \\ -\frac{L}{2} \\ 0 \end{cases}$$

$$\begin{cases} \dot{x}_{B0} \\ \dot{y}_{B0} \end{cases} = [\mathcal{Q}_x \dot{x}_w + \mathcal{Q}_y \dot{y}_w + \mathcal{Q}_q \dot{\mathbf{q}}_w] A_{01} \begin{cases} 1 \\ \frac{L}{2} \\ 0 \end{cases} \quad (22)$$

$$\text{Where: } \mathcal{Q}_x = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3 \times 3}, \mathcal{Q}_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{3 \times 3}, \mathcal{Q}_q = \begin{bmatrix} 0 & 0 & 0 \\ y_w & 0 & -1 \\ -x_w & 1 & 0 \end{bmatrix}_{3 \times 3}$$

From (21) and (22), we have:

$$\begin{cases} 1 \\ -\frac{L}{2} \\ 0 \end{cases}^T A_{01}^T [\mathcal{Q}_x^T \dot{x}_w + \mathcal{Q}_y^T \dot{y}_w + \mathcal{Q}_q^T \dot{\mathbf{q}}_w] A_{01} \begin{cases} 0 \\ -L \\ 0 \end{cases} = 0$$

$$\begin{cases} 1 \\ \frac{L}{2} \\ 0 \end{cases}^T A_{01}^T [\mathcal{Q}_x^T \dot{x}_w + \mathcal{Q}_y^T \dot{y}_w + \mathcal{Q}_q^T \dot{\mathbf{q}}_w] A_{01} \begin{cases} 0 \\ -L \\ 0 \end{cases} = 0 \quad (23)$$

Both of them lead to the following relation:

$$\cos \mathbf{q}_w dx_w + \sin \mathbf{q}_w dy_w = 0, \text{ or } \frac{\dot{y}_w}{\dot{x}_w} = -\frac{\cos \mathbf{q}_w}{\sin \mathbf{q}_w} \quad (24)$$

B. For the caster wheel, the location of the center of the wheel can be represented as:

$$\begin{cases} 1 \\ x_{lc0} \\ y_{lc0} \end{cases} = A_{01} A_{12} \begin{cases} 1 \\ L_c \\ 0 \end{cases} \quad (25)$$

Then its velocity can be represented as:

$$\begin{aligned} \vec{V}_{lc0} &= \dot{x}_{lc0} \vec{i} + \dot{y}_{lc0} \vec{j} \\ &= [\dot{x}_w - x_c \sin \mathbf{q}_w \dot{\mathbf{q}}_w - y_c \cos \mathbf{q}_w \dot{\mathbf{q}}_w + L_c \cos(\mathbf{q}_c - \mathbf{q}_w) (\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w)] \vec{i} \\ &\quad + [\dot{y}_w + x_c \cos \mathbf{q}_w \dot{\mathbf{q}}_w - y_c \sin \mathbf{q}_w \dot{\mathbf{q}}_w - L_c \sin(\mathbf{q}_c - \mathbf{q}_w) (\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w)] \vec{j} \end{aligned} \quad (26)$$

The displacement vector \vec{r}_{hc} from the hinge joint to the center of the wheel in global coordinate system can be got from:

$$\begin{cases} 1 \\ x_{lc0} \\ y_{lc0} \end{cases} - \begin{cases} 1 \\ x_{c0} \\ y_{c0} \end{cases} = A_{01} A_{12} \begin{cases} 1 \\ L_c \\ 0 \end{cases} - A_{01} \begin{cases} 1 \\ x_c \\ y_c \end{cases} \quad (27)$$

$$\text{So } r_{hc} = x_{hc} \vec{i} + y_{hc} \vec{j} = L_c \sin(\mathbf{q}_c - \mathbf{q}_w) \vec{i} + L_c \cos(\mathbf{q}_c - \mathbf{q}_w) \vec{j} \quad (28)$$

For the caster wheel constraint, there exists such relation: the velocity of the center of the wheel \vec{V}_{lc0} in this 2D plan should be parallel to the displacement vector \vec{r}_{hc} of the center of the caster wheel to its rotation joint on the walker frame:

$$\frac{\dot{y}_{lc0}}{\dot{x}_{lc0}} = \frac{y_{hc}}{x_{hc}} \quad (29)$$

Hence we get the second kinematic constraint equation for the system:

$$\sin(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w - \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w + (x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c) \dot{\mathbf{q}}_w + L_c (\dot{\mathbf{q}}_w - \dot{\mathbf{q}}_c) = 0 \quad (30)$$

which is due to the nonholonomic constraint from the caster wheel.

So (24) and (30) form the nonholonomic constraint equations for this system.

General forces for the system

To get the general forces for the system, the actual forces/moments added on the walker system in this planar case are: forces and moments exerted on the two handles of the walker, friction forces on the three wheels and the forces due to nonholonomic constraints.

In order to solve for the friction forces on the three wheels, we assume that the three wheels of the walker always contact the ground. Thus, we can get reaction force in vertical direction using the equilibrium principle. There are three static equations:

$$\left\{ \begin{array}{l} N_1 + N_2 + N_3 + F_{z1} + F_{z2} = mg \\ F_{z1} y_1 + F_{z2} y_1 - mg v_m + N_1 (d + L_c \cos \mathbf{q}_c) + T_{x1} + T_{x2} - (F_{y1} + F_{y2}) h = 0 \\ -F_{z1} x_1 - F_{z2} (L - x_1) - N_1 \left(\frac{L}{2} + L_c \sin \mathbf{q}_c \right) + mg u_m - N_3 L + T_{y1} + T_{y2} + (F_{x1} + F_{x2}) h = 0 \end{array} \right. \quad (31)$$

Then the vertical reaction forces are:

$$\begin{aligned}
\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} &= \begin{Bmatrix} \frac{-(F_{z1} + F_{z2})y_1 + mgv_m + (F_{y1} + F_{y2})h - (T_{x1} + T_{x2})}{d + L_c \cos \mathbf{q}_c}; \\ - (F_{z1} + F_{z2}) + mg - \frac{-(F_{z1} + F_{z2})y_1 + mgv_m + (F_{y1} + F_{y2})h - (T_{x1} + T_{x2})}{d + L_c \cos \mathbf{q}_c}; \\ \left. \begin{aligned} & - F_{z1}x_1 - F_{z2}(L - x_1) - \frac{\left(\frac{L}{2} + L_c \sin \mathbf{q}_c\right)}{d + L_c \cos \mathbf{q}_c} \left[-(F_{z1} + F_{z2})y_1 + (F_{y1} + F_{y2})h + mgv_m - (T_{x1} + T_{x2}) \right] / L; \\ & + mgu_m + (T_{y1} + T_{y2}) + (F_{x1} + F_{x2})h \end{aligned} \right\} \\ \left. \begin{aligned} & - F_{z1}x_1 - F_{z2}(L - x_1) - \frac{\left(\frac{L}{2} + L_c \sin \mathbf{q}_c\right)}{d + L_c \cos \mathbf{q}_c} \left[-(F_{z1} + F_{z2})y_1 + (F_{y1} + F_{y2})h + mgv_m - (T_{x1} + T_{x2}) \right] / L \\ & + mgu_m + (T_{y1} + T_{y2}) + (F_{x1} + F_{x2})h \end{aligned} \right\} / L \end{Bmatrix}
\end{aligned} \tag{32}$$

where: $F_{x1}, F_{y1}, F_{z1}, T_{x1}, T_{y1}, F_{x2}, F_{y2}, F_{z2}, T_{x2}, T_{y2}$ are signals from the two load cells. L is the distance between the two back wheels. d is the distance of the caster to the axis of the back wheels. h is the height from the load cell to the ground. L_c is the projection length to the x-y plan of the distance from the center of the caster wheel to its hinge joint with the frame. (u_m, v_m) are the position of the center of mass of the walker, m is the mass of the walker. (x_l, y_l) are the position of the left load cell relative to the center of left back wheel. All the coordinates are expressed in the frame's local coordinate system.

To identify the direction of the friction, we need to know the velocity of the wheel axis relative to its the local coordinate system.

For point 1:

$$\vec{r}_1 = \begin{bmatrix} x_w + x_c \cos \mathbf{q}_w - y_c \sin \mathbf{q}_w & \sin(\mathbf{q}_c - \mathbf{q}_w) & -\cos(\mathbf{q}_c - \mathbf{q}_w) \\ y_w + x_c \sin \mathbf{q}_w + y_c \cos \mathbf{q}_w & \cos(\mathbf{q}_c - \mathbf{q}_w) & \sin(\mathbf{q}_c - \mathbf{q}_w) \end{bmatrix} \begin{Bmatrix} 1 \\ L_c \\ 0 \end{Bmatrix} \tag{33}$$

then

$$\vec{V}_1 = \begin{bmatrix} \dot{x}_w - x_c \sin \mathbf{q}_w \dot{\mathbf{q}}_w - y_c \cos \mathbf{q}_w \dot{\mathbf{q}}_w & \cos(\mathbf{q}_c - \mathbf{q}_w)(\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) & \sin(\mathbf{q}_c - \mathbf{q}_w)(\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) \\ \dot{y}_w + x_c \cos \mathbf{q}_w \dot{\mathbf{q}}_w - y_c \sin \mathbf{q}_w \dot{\mathbf{q}}_w & -\sin(\mathbf{q}_c - \mathbf{q}_w)(\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) & \cos(\mathbf{q}_c - \mathbf{q}_w)(\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) \end{bmatrix} \begin{Bmatrix} 1 \\ L_c \\ 0 \end{Bmatrix}, \tag{34}$$

then the projection to the local coordinate system 2 is:

$$\vec{V}'_1 = \left(\begin{bmatrix} \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{bmatrix} \sin \mathbf{q}_c & -\cos \mathbf{q}_c \\ \cos \mathbf{q}_c & \sin \mathbf{q}_c \end{bmatrix} \right)^{-1} \times$$

$$\begin{bmatrix} \dot{x}_w - x_c \sin \mathbf{q}_w \dot{\mathbf{q}}_w - y_c \cos \mathbf{q}_w \dot{\mathbf{q}}_w & \cos(\mathbf{q}_c - \mathbf{q}_w)(\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) & \sin(\mathbf{q}_c - \mathbf{q}_w)(\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) \\ \dot{y}_w + x_c \cos \mathbf{q}_w \dot{\mathbf{q}}_w - y_c \sin \mathbf{q}_w \dot{\mathbf{q}}_w & -\sin(\mathbf{q}_c - \mathbf{q}_w)(\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) & \cos(\mathbf{q}_c - \mathbf{q}_w)(\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) \end{bmatrix} \begin{bmatrix} 1 \\ L_c \\ 0 \end{bmatrix}$$
(35)

In this case the direction of the friction is determined in x' direction:

$$F_{fric1} = -\text{sign}(\sin(\mathbf{q}_c - \mathbf{q}_w)\dot{x}_w + \cos(\mathbf{q}_c - \mathbf{q}_w)\dot{y}_w + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c)\dot{\mathbf{q}}_w) \mathbf{m}N_1 \quad (36)$$

For point 2 (A):

$$\vec{r}_2 = \begin{bmatrix} x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{L}{2} \\ 0 \end{bmatrix} \quad (37)$$

$$\text{then } \vec{V}_2 = \begin{bmatrix} \dot{x}_w & -\sin \mathbf{q}_w \dot{\mathbf{q}}_w & -\cos \mathbf{q}_w \dot{\mathbf{q}}_w \\ \dot{y}_w & \cos \mathbf{q}_w \dot{\mathbf{q}}_w & -\sin \mathbf{q}_w \dot{\mathbf{q}}_w \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{L}{2} \\ 0 \end{bmatrix} \quad (38)$$

then the projection to the local coordinate system 1 is:

$$\vec{V}'_2 = \begin{bmatrix} \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix}^{-1} \begin{bmatrix} \dot{x}_w & -\sin \mathbf{q}_w \dot{\mathbf{q}}_w & -\cos \mathbf{q}_w \dot{\mathbf{q}}_w \\ \dot{y}_w & \cos \mathbf{q}_w \dot{\mathbf{q}}_w & -\sin \mathbf{q}_w \dot{\mathbf{q}}_w \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{L}{2} \\ 0 \end{bmatrix} \quad (39)$$

In this case the direction of the friction is determined in y' direction:

$$F_{fric2} = -\text{sign}\left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w\right) \mathbf{m}N_2 \quad (40)$$

Similarly, for point 3(B):

$$F_{fric3} = -\text{sign}\left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w\right) \mathbf{m}N_3 \quad (41)$$

Also consider into the system the friction torque on the rotational joint connecting the two rigid bodies, which is chosen to be approximated as to be linearly dependent on the relative rotational velocity.

Then the virtual work done by these forces/moments are:

$$\begin{aligned}
dW &= \begin{Bmatrix} F_{x10} \\ F_{y10} \end{Bmatrix} \cdot d\bar{\mathbf{r}}_{lc1} + \begin{Bmatrix} F_{x20} \\ F_{y20} \end{Bmatrix} \cdot d\bar{\mathbf{r}}_{lc2} + (T_{z1} + T_{z2}) \cdot d\mathbf{q}_w + \bar{F}_{fric1} \cdot d\bar{\mathbf{r}}_1 + \bar{F}_{fric2} \cdot d\bar{\mathbf{r}}_2 + \bar{F}_{fric3} \cdot d\bar{\mathbf{r}}_3 + \bar{T}_{fric4} \cdot d\mathbf{q}_c \\
&= \begin{Bmatrix} F_{x1} \\ F_{y1} \end{Bmatrix}^T \begin{bmatrix} \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix}^T \cdot \mathbf{d} \begin{bmatrix} x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{Bmatrix} 1 \\ -x_{lc} \\ y_{lc} \end{Bmatrix} \\
&+ \begin{Bmatrix} F_{x2} \\ F_{y2} \end{Bmatrix}^T \begin{bmatrix} \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix}^T \cdot \mathbf{d} \begin{bmatrix} x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{Bmatrix} 1 \\ x_{lc} \\ y_{lc} \end{Bmatrix} \\
&+ (T_{z1} + T_{z2}) \cdot d\mathbf{q}_w + \begin{Bmatrix} -\text{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \\ 0 \end{pmatrix} \\ \mathbf{m}N_1 \end{Bmatrix}^T \left(\begin{bmatrix} \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{bmatrix} \sin \mathbf{q}_c & -\cos \mathbf{q}_c \\ \cos \mathbf{q}_c & \sin \mathbf{q}_c \end{bmatrix} \right)^T \\
&\cdot \mathbf{d} \begin{bmatrix} x_w + x_c \cos \mathbf{q}_w - y_c \sin \mathbf{q}_w & \sin(\mathbf{q}_c - \mathbf{q}_w) & -\cos(\mathbf{q}_c - \mathbf{q}_w) \\ y_w + x_c \sin \mathbf{q}_w + y_c \cos \mathbf{q}_w & \cos(\mathbf{q}_c - \mathbf{q}_w) & \sin(\mathbf{q}_c - \mathbf{q}_w) \end{bmatrix} \begin{Bmatrix} 1 \\ L_c \\ 0 \end{Bmatrix} \\
&+ \begin{Bmatrix} 0 \\ -\text{sign} \begin{pmatrix} -\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w \\ -\frac{L}{2} \dot{\mathbf{q}}_w \end{pmatrix} \\ \mathbf{m}N_2 \end{Bmatrix}^T \begin{bmatrix} \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix}^T \cdot \mathbf{d} \begin{bmatrix} x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{Bmatrix} 1 \\ -\frac{L}{2} \\ 0 \end{Bmatrix} \\
&+ \begin{Bmatrix} 0 \\ -\text{sign} \begin{pmatrix} -\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w \\ +\frac{L}{2} \dot{\mathbf{q}}_w \end{pmatrix} \\ \mathbf{m}N_3 \end{Bmatrix}^T \begin{bmatrix} \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix}^T \cdot \mathbf{d} \begin{bmatrix} x_w & \cos \mathbf{q}_w & -\sin \mathbf{q}_w \\ y_w & \sin \mathbf{q}_w & \cos \mathbf{q}_w \end{bmatrix} \begin{Bmatrix} 1 \\ \frac{L}{2} \\ 0 \end{Bmatrix} \\
&- k \dot{\mathbf{q}}_c d\mathbf{q}_c
\end{aligned} \tag{42}$$

where: $F_{x1}, F_{y1}, F_{z1}, T_{x1}, T_{y1}, T_{z1}, F_{x2}, F_{y2}, F_{z2}, T_{x2}, T_{y2}, T_{z2}$ are the signals got from the two load cells, the two load cells are mounted at the point $(-x_{lc}, y_{lc})$ and (x_{lc}, y_{lc}) in the local coordinate system on the rigid body 1 (walker frame).

The general constraint forces corresponding to the two nonholonomic constraints can be expressed as^[3]:

$$C_i = \sum_{j=1}^2 \mathbf{I}_j a_{ji} \quad (i=1,2,\dots,4) \tag{43}$$

where \mathbf{I}_j is the corresponding LaGrange multiplier.

From (24):

$$\begin{cases} a_{11} = \cos \mathbf{q}_w \\ a_{12} = \sin \mathbf{q}_w \\ a_{13} = 0 \\ a_{14} = 0 \end{cases} \quad (44)$$

From (30):

$$\begin{cases} a_{21} = -\cos(\mathbf{q}_c - \mathbf{q}_w) \\ a_{22} = \sin(\mathbf{q}_c - \mathbf{q}_w) \\ a_{23} = x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c \\ a_{24} = -L_c \end{cases} \quad (45)$$

Then the general nonholonomic forces are:

$$\begin{cases} C_1 \\ C_2 \\ C_3 \\ C_4 \end{cases} = \begin{cases} \mathbf{I}_1 \cos \mathbf{q}_w - \mathbf{I}_2 \cos(\mathbf{q}_c - \mathbf{q}_w) \\ \mathbf{I}_1 \sin \mathbf{q}_w + \mathbf{I}_2 \sin(\mathbf{q}_c - \mathbf{q}_w) \\ \mathbf{I}_2 (x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c) \\ -\mathbf{I}_2 L_c \end{cases} \quad (46)$$

Then the total general forces can be expressed as:

$$\begin{aligned}
Q_1 &= \frac{\partial dW}{\partial d\mathbf{x}_w} + C_1 = F_{x1} \cos \mathbf{q}_w - F_{y1} \sin \mathbf{q}_w + F_{x2} \cos \mathbf{q}_w - F_{y2} \sin \mathbf{q}_w \\
&\quad - \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) mN_1 \sin(\mathbf{q}_c - \mathbf{q}_w) \\
&\quad + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) mN_2 \sin \mathbf{q}_w \\
&\quad + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) mN_3 \sin \mathbf{q}_w \\
&\quad + I_1 \cos \mathbf{q}_w - I_2 \cos(\mathbf{q}_c - \mathbf{q}_w) \\
Q_2 &= \frac{\partial dW}{\partial dy_w} + C_2 = F_{x1} \sin \mathbf{q}_w + F_{y1} \cos \mathbf{q}_w + F_{x2} \sin \mathbf{q}_w + F_{y2} \cos \mathbf{q}_w \\
&\quad - \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) mN_1 \cos(\mathbf{q}_c - \mathbf{q}_w) \\
&\quad - \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) mN_2 \cos \mathbf{q}_w \\
&\quad - \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) mN_3 \cos \mathbf{q}_w \\
&\quad + I_1 \sin \mathbf{q}_w + I_2 \sin(\mathbf{q}_c - \mathbf{q}_w) \\
Q_3 &= \frac{\partial dW}{\partial d\mathbf{q}_w} + C_3 = -x_{lc} F_{y1} - y_{lc} F_{x1} + x_{lc} F_{y2} - y_{lc} F_{x2} + T_{z1} + T_{z2} \\
&\quad + \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) mN_1 (y_c \sin \mathbf{q}_c - x_c \cos \mathbf{q}_c) \\
&\quad + m \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ - \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \frac{L}{2} \\
&\quad + I_2 (x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c) \\
Q_4 &= \frac{\partial dW}{\partial d\mathbf{q}_c} + C_4 = -I_2 L_c - k \dot{\mathbf{q}}_c
\end{aligned} \tag{47}$$

4. Planar dynamics for the walker

Synthesize all the content derived above; this model contained 4 general coordinates

$\begin{Bmatrix} x_w \\ y_w \\ \mathbf{q}_w \\ \mathbf{q}_c \end{Bmatrix}$ and two unknown LaGrange multipliers $\begin{Bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{Bmatrix}$ due to the nonholonomic constraints

on the walker. So there are totally six unknown variables in this model. The governing differential equations are:

$$[M_s] \{\ddot{\mathbf{x}}\} + C(\bar{\mathbf{x}}, \dot{\mathbf{x}}) = \{Q\} \quad (48)$$

where: $[M_s]$, $C(\bar{\mathbf{x}}, \dot{\mathbf{x}})$ and $\{Q\}$ are given out at (16), (17) and (47), respectively. There are 4 equations in (48), together with the two nonholonomic constraint equations shown at (24) and (30), we can get the unique solution of the six variables in this model. Thus we can access the total information about the dynamic movement and constraint forces that we need to know to investigate user intent and further control issues.

5. Transform the model into standard format

To do the simulation, first, the two LaGrange multipliers are needed to be taken off from the equations.

To do that, we rewrite the fourth equation of (48) as:

$$\mathbf{I}_2 = -\frac{1}{L_c} [M_{41}\ddot{x}_w + M_{42}\ddot{y}_w + M_{43}\ddot{\mathbf{q}}_w + M_{44}\ddot{\mathbf{q}}_c + C(4) + k\dot{\mathbf{q}}_c] \quad (49)$$

substitute into the first equation of (48):

$$\mathbf{l}_1 = \frac{1}{\cos \mathbf{q}_w} \left\{ \begin{array}{l} M_{11}\ddot{x}_w + M_{12}\ddot{y}_w + M_{13}\ddot{\mathbf{q}}_w + M_{14}\ddot{\mathbf{q}}_c + C(1) \\ -\frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} [M_{41}\ddot{x}_w + M_{42}\ddot{y}_w + M_{43}\ddot{\mathbf{q}}_w + M_{44}\ddot{\mathbf{q}}_c + C(4) + k\dot{\mathbf{q}}_c] \\ -F_{x1} \cos \mathbf{q}_w + F_{y1} \sin \mathbf{q}_w - F_{x2} \cos \mathbf{q}_w + F_{y2} \sin \mathbf{q}_w \\ + \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w)\dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w)\dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c)\dot{\mathbf{q}}_w \end{array} \right) mN_1 \sin(\mathbf{q}_c - \mathbf{q}_w) \\ -m \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \sin \mathbf{q}_w \end{array} \right\} \quad (50)$$

substitute (49), (50) into the third and second equations of (48), respectively:

$$\begin{aligned}
& M_{31}\ddot{x}_w + M_{32}\ddot{y}_w + M_{33}\ddot{\mathbf{q}}_w + M_{34}\ddot{\mathbf{q}}_c + C(3) = \\
& -x_{lc}F_{y1} - y_{lc}F_{x1} + x_{lc}F_{y2} - y_{lc}F_{x2} + T_{z1} + T_{z2} + \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w)\dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w)\dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c)\dot{\mathbf{q}}_w \end{array} \right) mN_1 (y_c \sin \mathbf{q}_c - x_c \cos \mathbf{q}_c) \\
& + m \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ - \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \frac{L}{2} \\
& - \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} [M_{41}\ddot{x}_w + M_{42}\ddot{y}_w + M_{43}\ddot{\mathbf{q}}_w + M_{44}\ddot{\mathbf{q}}_c + C(4) + k\dot{\mathbf{q}}_c]
\end{aligned} \quad (51)$$

$$\begin{aligned}
& M_{21}\ddot{x}_w + M_{22}\ddot{y}_w + M_{23}\ddot{\mathbf{q}}_w + M_{24}\ddot{\mathbf{q}}_c + C(2) = \\
& F_{x1} \sin \mathbf{q}_w + F_{y1} \cos \mathbf{q}_w + F_{x2} \sin \mathbf{q}_w + F_{y2} \cos \mathbf{q}_w - \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) \mathbf{m} N_1 \cos(\mathbf{q}_c - \mathbf{q}_w) \\
& - \mathbf{m} \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \cos \mathbf{q}_w \\
& + \frac{\sin \mathbf{q}_w}{\cos \mathbf{q}_w} \left\{ \begin{array}{l} M_{11}\ddot{x}_w + M_{12}\ddot{y}_w + M_{13}\ddot{\mathbf{q}}_w + M_{14}\ddot{\mathbf{q}}_c + C(1) \\ - \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} [M_{41}\ddot{x}_w + M_{42}\ddot{y}_w + M_{43}\ddot{\mathbf{q}}_w + M_{44}\ddot{\mathbf{q}}_c + C(4) + k\dot{\mathbf{q}}_c] \\ - F_{x1} \cos \mathbf{q}_w + F_{y1} \sin \mathbf{q}_w - F_{x2} \cos \mathbf{q}_w + F_{y2} \sin \mathbf{q}_w \\ + \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) \mathbf{m} N_1 \sin(\mathbf{q}_c - \mathbf{q}_w) \\ - \mathbf{m} \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \sin \mathbf{q}_w \end{array} \right\} \\
& - \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} [M_{41}\ddot{x}_w + M_{42}\ddot{y}_w + M_{43}\ddot{\mathbf{q}}_w + M_{44}\ddot{\mathbf{q}}_c + C(4) + k\dot{\mathbf{q}}_c]
\end{aligned} \tag{52}$$

simplify (51) and (52), we get:

$$\begin{aligned}
& \left(M_{31} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{41} \right) \ddot{\mathbf{x}}_w + \left(M_{32} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{42} \right) \ddot{\mathbf{y}}_w \\
& + \left(M_{33} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{43} \right) \ddot{\mathbf{q}}_w + \left(M_{34} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{44} \right) \ddot{\mathbf{q}}_c \\
& + C(3) + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} C(4) = -x_{lc} F_{y1} - y_{lc} F_{x1} + x_{lc} F_{y2} - y_{lc} F_{x2} + T_{z1} + T_{z2} \\
& + \mathbf{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{\mathbf{x}}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{\mathbf{y}}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{pmatrix} \mathbf{m} \mathbf{V}_1 (y_c \sin \mathbf{q}_c - x_c \cos \mathbf{q}_c) \\
& + \mathbf{m} \begin{pmatrix} \mathbf{sign} \left(-\sin \mathbf{q}_w \dot{\mathbf{x}}_w + \cos \mathbf{q}_w \dot{\mathbf{y}}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ - \mathbf{sign} \left(-\sin \mathbf{q}_w \dot{\mathbf{x}}_w + \cos \mathbf{q}_w \dot{\mathbf{y}}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \frac{L}{2} - \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} k \dot{\mathbf{q}}_c \quad (53)
\end{aligned}$$

$$\begin{aligned}
& \left[M_{21} - tg \mathbf{q}_w M_{11} + \frac{tg \mathbf{q}_w \cos(\mathbf{q}_c - \mathbf{q}_w) + \sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{41} \right] \ddot{\mathbf{x}}_w \\
& + \left[M_{22} - tg \mathbf{q}_w M_{12} + \frac{tg \mathbf{q}_w \cos(\mathbf{q}_c - \mathbf{q}_w) + \sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{42} \right] \ddot{\mathbf{y}}_w \\
& + \left[M_{23} - tg \mathbf{q}_w M_{13} + \frac{tg \mathbf{q}_w \cos(\mathbf{q}_c - \mathbf{q}_w) + \sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{43} \right] \ddot{\mathbf{q}}_w \\
& + \left[M_{24} - tg \mathbf{q}_w M_{14} + \frac{tg \mathbf{q}_w \cos(\mathbf{q}_c - \mathbf{q}_w) + \sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{44} \right] \ddot{\mathbf{q}}_c \\
& + C(2) - tg \mathbf{q}_w C(1) + \frac{tg \mathbf{q}_w \cos(\mathbf{q}_c - \mathbf{q}_w) + \sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} C(4) = \\
& \frac{F_{y1} + F_{y2}}{\cos \mathbf{q}_w} + \mathbf{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{\mathbf{x}}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{\mathbf{y}}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{pmatrix} \mathbf{m} \mathbf{V}_1 [tg \mathbf{q}_w \sin(\mathbf{q}_c - \mathbf{q}_w) - \cos(\mathbf{q}_c - \mathbf{q}_w)] \\
& \mathbf{m} \begin{pmatrix} \mathbf{sign} \left(-\sin \mathbf{q}_w \dot{\mathbf{x}}_w + \cos \mathbf{q}_w \dot{\mathbf{y}}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \mathbf{sign} \left(-\sin \mathbf{q}_w \dot{\mathbf{x}}_w + \cos \mathbf{q}_w \dot{\mathbf{y}}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \frac{L}{2} - \frac{tg \mathbf{q}_w \cos(\mathbf{q}_c - \mathbf{q}_w) + \sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} k \dot{\mathbf{q}}_c \quad (54)
\end{aligned}$$

In order to avoid the singularity of the ODE group in the solution, (24) and (30) are derived once more:

$$\cos \mathbf{q}_w \ddot{x}_w + \sin \mathbf{q}_w \ddot{y}_w = \sin \mathbf{q}_w \dot{\mathbf{q}}_w \dot{x}_w - \cos \mathbf{q}_w \dot{\mathbf{q}}_w \dot{y}_w \quad (55)$$

$$\begin{aligned} & -\cos(\mathbf{q}_c - \mathbf{q}_w) \ddot{x}_w + \sin(\mathbf{q}_c - \mathbf{q}_w) \ddot{y}_w + (x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c) \ddot{\mathbf{q}}_w - L_c \ddot{\mathbf{q}}_c = \\ & -\sin(\mathbf{q}_c - \mathbf{q}_w) (\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) \dot{x}_w - \cos(\mathbf{q}_c - \mathbf{q}_w) (\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) \dot{y}_w + (y_c \sin \mathbf{q}_c - x_c \cos \mathbf{q}_c) \dot{\mathbf{q}}_c \dot{\mathbf{q}}_w \end{aligned} \quad (56)$$

Change the modified ODE group (53), (54), (55) and (56) into the following format:

$$A \dot{x} = B \quad (57)$$

$$\text{where: } A = \begin{bmatrix} [I_{4 \times 4}] & [0]_{4 \times 4} \\ [0]_{4 \times 4} & [SA]_{4 \times 4} \end{bmatrix}_{8 \times 8},$$

$$SA_{11} = M_{31} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{41}$$

$$SA_{12} = M_{32} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{42}$$

$$SA_{13} = M_{33} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{43}$$

$$SA_{14} = M_{34} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{44}$$

$$SA_{21} = M_{21} - tg \mathbf{q}_w M_{11} + \frac{tg \mathbf{q}_w \cos(\mathbf{q}_c - \mathbf{q}_w) + \sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{41}$$

$$SA_{22} = M_{22} - tg \mathbf{q}_w M_{12} + \frac{tg \mathbf{q}_w \cos(\mathbf{q}_c - \mathbf{q}_w) + \sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{42}$$

$$SA_{23} = M_{23} - tg \mathbf{q}_w M_{13} + \frac{tg \mathbf{q}_w \cos(\mathbf{q}_c - \mathbf{q}_w) + \sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{43}$$

$$SA_{24} = M_{24} - tg \mathbf{q}_w M_{14} + \frac{tg \mathbf{q}_w \cos(\mathbf{q}_c - \mathbf{q}_w) + \sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{44}$$

$$SA_{31} = \cos \mathbf{q}_w$$

$$SA_{32} = \sin \mathbf{q}_w$$

$$SA_{33} = SA_{34} = 0$$

$$SA_{41} = -\cos(\mathbf{q}_c - \mathbf{q}_w)$$

$$SA_{42} = \sin(\mathbf{q}_c - \mathbf{q}_w)$$

$$SA_{43} = x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c$$

$$SA_{44} = -L_c$$

(58)

$$\begin{aligned}
& \begin{matrix} \dot{x}_w; \\ \dot{y}_w; \\ \dot{\mathbf{q}}_w; \\ \dot{\mathbf{q}}_c; \end{matrix} \\
& -x_{lc}F_{y1} - y_{lc}F_{x1} + x_{lc}F_{y2} - y_{lc}F_{x2} + T_{z1} + T_{z2} \\
& + \text{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w)\dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w)\dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c)\dot{\mathbf{q}}_w \end{pmatrix} \mathbf{m}N_1(y_c \sin \mathbf{q}_c - x_c \cos \mathbf{q}_c) \\
& + \mathbf{m} \begin{pmatrix} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ - \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \left\{ \frac{L}{2} - C(3) - \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} [C(4) + k\dot{\mathbf{q}}_c] \right\} \\
& B = \left\{ \begin{aligned} & \frac{F_{y1} + F_{y2}}{\cos \mathbf{q}_w} + \text{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w)\dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w)\dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c)\dot{\mathbf{q}}_w \end{pmatrix} \mathbf{m}N_1 [tg \mathbf{q}_w \sin(\mathbf{q}_c - \mathbf{q}_w) - \cos(\mathbf{q}_c - \mathbf{q}_w)] \\ & \mathbf{m} \begin{pmatrix} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \\ & - \frac{\quad}{\cos \mathbf{q}_w} - C(2) + tg \mathbf{q}_w C(1) - \\ & \frac{tg \mathbf{q}_w \cos(\mathbf{q}_c - \mathbf{q}_w) + \sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} [C(4) + k\dot{\mathbf{q}}_c]; \\ & \sin \mathbf{q}_w \dot{\mathbf{q}}_w \dot{x}_w - \cos \mathbf{q}_w \dot{\mathbf{q}}_w \dot{y}_w; \\ & - \sin(\mathbf{q}_c - \mathbf{q}_w)(\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w)\dot{x}_w - \cos(\mathbf{q}_c - \mathbf{q}_w)(\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w)\dot{y}_w + (y_c \sin \mathbf{q}_c - x_c \cos \mathbf{q}_c)\dot{\mathbf{q}}_c \dot{\mathbf{q}}_w \end{aligned} \right\} \\
& \hspace{15em} (59)
\end{aligned}$$

while: $x = \{x_w \ y_w \ \mathbf{q}_w \ \mathbf{q}_c \ \dot{x}_w \ \dot{y}_w \ \dot{\mathbf{q}}_w \ \dot{\mathbf{q}}_c\}^T$

Initial conditions must comply with (24) and (30) to get correct solution.

$\dot{x} = A^{-1}B$, use *ode45* in matlab to solve.

6. Further discussions

In this model, the forces to make nonholonomic constraint exist are represented by the LaGrange multipliers I_1 and I_2 . The physical meaning of these LaGrange multipliers is:

I_1 : The force applied on the two back wheels on their wheel axis direction. Expressed in local coordinate, positive direction is to the right.

I_2 : The negative of the force applied on the frontal caster wheel on the perpendicular axis of its moving direction. Expressed in local coordinate, positive direction is to the right.

The condition that nonholonomic constraints exist is: the constraint force I_1 or I_2 is less than its corresponding limit: the corresponding maximal frictional force. If the required constraint force I_1 or I_2 is greater than its maximal friction the ground can provide, then the corresponding nonholonomic constraint will no longer exist, the corresponding wheel will slide along the direction of the constraints force.

If this case happens, then we need to update the governing differential equations by setting the corresponding constraint force I_i to be the maximal frictional force and discard the corresponding constraint equation from the equation group. By reducing the same amount of variables and constraints equations, we still get the same amount of variable as differential equations, thus we still get a unique solution of the system's dynamic response.

In general, there will be four possible cases happening when all the wheels have contact with the ground:

1. All nonholonomic constraints exist
2. Only frontal wheel nonholonomic constraint exist
3. Only back wheels nonholonomic constraint exist
4. All nonholonomic constraints don't exist

1. For the first case, the standard format of the equations are (57), (58) and (59).

2. The second case happens when $I_1 > \mathbf{m}_n(N_2 + N_3)$ or $I_1 < -\mathbf{m}_n(N_2 + N_3)$, then take $I_1 = \mathbf{m}_n(N_2 + N_3)$ or $I_1 = -\mathbf{m}_n(N_2 + N_3)$.

Use (49), (51) and the following equation:

$$\begin{aligned}
M_{11}\ddot{x}_w + M_{12}\ddot{y}_w + M_{13}\ddot{\mathbf{q}}_w + M_{14}\ddot{\mathbf{q}}_c + C(1) &= F_{x1} \cos \mathbf{q}_w - F_{y1} \sin \mathbf{q}_w + F_{x2} \cos \mathbf{q}_w - F_{y2} \sin \mathbf{q}_w \\
- \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) \mathbf{m} N_1 \sin(\mathbf{q}_c - \mathbf{q}_w) + \mathbf{m} \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \sin \mathbf{q}_w \\
+ I_1 \cos \mathbf{q}_w + \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} [M_{41}\ddot{x}_w + M_{42}\ddot{y}_w + M_{43}\ddot{\mathbf{q}}_w + M_{44}\ddot{\mathbf{q}}_c + C(4) + k\dot{\mathbf{q}}_c]
\end{aligned} \tag{60}$$

$$\begin{aligned}
& M_{21}\ddot{x}_w + M_{22}\ddot{y}_w + M_{23}\ddot{\mathbf{q}}_w + M_{24}\ddot{\mathbf{q}}_c + C(2) = \\
& F_{x1} \sin \mathbf{q}_w + F_{y1} \cos \mathbf{q}_w + F_{x2} \sin \mathbf{q}_w + F_{y2} \cos \mathbf{q}_w - \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) \mathbf{m} N_1 \cos(\mathbf{q}_c - \mathbf{q}_w) \\
& - \mathbf{m} \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \cos \mathbf{q}_w \\
& + \mathbf{I}_1 \sin \mathbf{q}_w - \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} [M_{41}\ddot{x}_w + M_{42}\ddot{y}_w + M_{43}\ddot{\mathbf{q}}_w + M_{44}\ddot{\mathbf{q}}_c + C(4) + k\dot{\mathbf{q}}_c]
\end{aligned} \tag{61}$$

Simplify (60), (61), we have:

$$\begin{aligned}
& \left(M_{11} - \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{41} \right) \ddot{x}_w + \left(M_{12} - \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{42} \right) \ddot{y}_w + \left(M_{13} - \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{43} \right) \ddot{\mathbf{q}}_w \\
& + \left(M_{14} - \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{44} \right) \ddot{\mathbf{q}}_c = F_{x1} \cos \mathbf{q}_w - F_{y1} \sin \mathbf{q}_w + F_{x2} \cos \mathbf{q}_w - F_{y2} \sin \mathbf{q}_w \\
& - \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) \mathbf{m} N_1 \sin(\mathbf{q}_c - \mathbf{q}_w) + \mathbf{m} \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \sin \mathbf{q}_w \\
& + \mathbf{I}_1 \cos \mathbf{q}_w + \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} [C(4) + k\dot{\mathbf{q}}_c] - C(1)
\end{aligned} \tag{62}$$

$$\begin{aligned}
& \left(M_{21} + \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{41} \right) \ddot{x}_w + \left(M_{22} + \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{42} \right) \ddot{y}_w + \left(M_{23} + \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{43} \right) \ddot{\mathbf{q}}_w \\
& + \left(M_{24} + \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{44} \right) \ddot{\mathbf{q}}_c = F_{x1} \sin \mathbf{q}_w + F_{y1} \cos \mathbf{q}_w + F_{x2} \sin \mathbf{q}_w + F_{y2} \cos \mathbf{q}_w \\
& - \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) \mathbf{m} N_1 \cos(\mathbf{q}_c - \mathbf{q}_w) \\
& - \mathbf{m} \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \cos \mathbf{q}_w \\
& + \mathbf{I}_1 \sin \mathbf{q}_w - \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} [C(4) + k\dot{\mathbf{q}}_c] - C(2)
\end{aligned} \tag{63}$$

(62), (63), (53) and (56) together give:

$$\begin{aligned}
SA_{11} &= M_{11} - \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{41} \\
SA_{12} &= M_{12} - \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{42} \\
SA_{13} &= M_{13} - \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{43} \\
SA_{14} &= M_{14} - \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{44} \\
SA_{21} &= M_{21} + \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{41} \\
SA_{22} &= M_{22} + \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{42} \\
SA_{23} &= M_{23} + \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{43} \\
SA_{24} &= M_{24} + \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} M_{44} \\
SA_{31} &= M_{31} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{41} \\
SA_{32} &= M_{32} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{42} \\
SA_{33} &= M_{33} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{43} \\
SA_{34} &= M_{34} + \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} M_{44} \\
SA_{41} &= -\cos(\mathbf{q}_c - \mathbf{q}_w) \\
SA_{42} &= \sin(\mathbf{q}_c - \mathbf{q}_w) \\
SA_{43} &= x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c \\
SA_{44} &= -L_c
\end{aligned} \tag{64}$$

$$\begin{aligned}
& \begin{matrix} \dot{x}_w; \\ \dot{y}_w; \\ \dot{\mathbf{q}}_w; \\ \dot{\mathbf{q}}_c; \end{matrix} \\
& F_{x1} \cos \mathbf{q}_w - F_{y1} \sin \mathbf{q}_w + F_{x2} \cos \mathbf{q}_w - F_{y2} \sin \mathbf{q}_w \\
& - \operatorname{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{pmatrix} \mathbf{m} N_1 \sin(\mathbf{q}_c - \mathbf{q}_w) + \mathbf{m} \begin{pmatrix} \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \sin \mathbf{q}_w \\
& + \mathbf{I}_1 \cos \mathbf{q}_w + \frac{\cos(\mathbf{q}_c - \mathbf{q}_w)}{L_c} [C(4) + k \dot{\mathbf{q}}_c] - C(1); \\
& \begin{matrix} F_{x1} \sin \mathbf{q}_w + F_{y1} \cos \mathbf{q}_w + F_{x2} \sin \mathbf{q}_w + F_{y2} \cos \mathbf{q}_w \\ - \operatorname{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{pmatrix} \mathbf{m} N_1 \cos(\mathbf{q}_c - \mathbf{q}_w) \\ - \mathbf{m} \begin{pmatrix} \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \cos \mathbf{q}_w \\ + \mathbf{I}_1 \sin \mathbf{q}_w - \frac{\sin(\mathbf{q}_c - \mathbf{q}_w)}{L_c} [C(4) + k \dot{\mathbf{q}}_c] - C(2); \end{matrix} \\
& - x_{lc} F_{y1} - y_{lc} F_{x1} + x_{lc} F_{y2} - y_{lc} F_{x2} + T_{z1} + T_{z2} \\
& + \operatorname{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{pmatrix} \mathbf{m} N_1 (y_c \sin \mathbf{q}_c - x_c \cos \mathbf{q}_c) \\
& + \mathbf{m} \begin{pmatrix} \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ - \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \left\{ \frac{L}{2} - C(3) - \frac{(x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)}{L_c} [C(4) + k \dot{\mathbf{q}}_c] \right\} \\
& \quad - \sin(\mathbf{q}_c - \mathbf{q}_w) (\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) \dot{x}_w - \cos(\mathbf{q}_c - \mathbf{q}_w) (\dot{\mathbf{q}}_c - \dot{\mathbf{q}}_w) \dot{y}_w + (y_c \sin \mathbf{q}_c - x_c \cos \mathbf{q}_c) \dot{\mathbf{q}}_c \dot{\mathbf{q}}_w
\end{aligned} \tag{65}$$

$$\dot{x} = A^{-1} B, \quad A^{-1} = \begin{bmatrix} [I_{4 \times 4}] & [0]_{4 \times 4} \\ [0]_{4 \times 4} & [SA]^{-1}_{4 \times 4} \end{bmatrix}_{8 \times 8}$$

3. The third case happens when $\mathbf{I}_2 > \mathbf{m}_n N_1$ or $\mathbf{I}_2 < -\mathbf{m}_n N_1$, then take $\mathbf{I}_2 = \mathbf{m}_n N_1$ or $\mathbf{I}_2 = -\mathbf{m}_n N_1$.

$$\text{Then: } \mathbf{l}_1 = \frac{1}{\cos \mathbf{q}_w} \left\{ \begin{array}{l} M_{11}\ddot{x}_w + M_{12}\ddot{y}_w + M_{13}\ddot{\mathbf{q}}_w + M_{14}\ddot{\mathbf{q}}_c + C(1) + \mathbf{l}_2 \cos(\mathbf{q}_c - \mathbf{q}_w) \\ - F_{x1} \cos \mathbf{q}_w + F_{y1} \sin \mathbf{q}_w - F_{x2} \cos \mathbf{q}_w + F_{y2} \sin \mathbf{q}_w \\ + \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) \mathbf{m} N_1 \sin(\mathbf{q}_c - \mathbf{q}_w) \\ - \mathbf{m} \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \sin \mathbf{q}_w \end{array} \right\} \quad (66)$$

$$M_{21}\ddot{x}_w + M_{22}\ddot{y}_w + M_{23}\ddot{\mathbf{q}}_w + M_{24}\ddot{\mathbf{q}}_c + C(2) =$$

$$F_{x1} \sin \mathbf{q}_w + F_{y1} \cos \mathbf{q}_w + F_{x2} \sin \mathbf{q}_w + F_{y2} \cos \mathbf{q}_w - \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) \mathbf{m} N_1 \cos(\mathbf{q}_c - \mathbf{q}_w)$$

$$- \mathbf{m} \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \cos \mathbf{q}_w$$

$$+ \frac{\sin \mathbf{q}_w}{\cos \mathbf{q}_w} \left\{ \begin{array}{l} M_{11}\ddot{x}_w + M_{12}\ddot{y}_w + M_{13}\ddot{\mathbf{q}}_w + M_{14}\ddot{\mathbf{q}}_c + C(1) + \mathbf{l}_2 \cos(\mathbf{q}_c - \mathbf{q}_w) \\ - F_{x1} \cos \mathbf{q}_w + F_{y1} \sin \mathbf{q}_w - F_{x2} \cos \mathbf{q}_w + F_{y2} \sin \mathbf{q}_w \\ + \text{sign} \left(\begin{array}{l} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{array} \right) \mathbf{m} N_1 \sin(\mathbf{q}_c - \mathbf{q}_w) \\ - \mathbf{m} \left(\begin{array}{l} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{array} \right) \sin \mathbf{q}_w \end{array} \right\} + \mathbf{l}_2 \sin(\mathbf{q}_c - \mathbf{q}_w)$$

(67)

$$\begin{aligned}
& M_{31}\ddot{x}_w + M_{32}\ddot{y}_w + M_{33}\ddot{\mathbf{q}}_w + M_{34}\ddot{\mathbf{q}}_c + C(3) = \\
& -x_{lc}F_{y1} - y_{lc}F_{x1} + x_{lc}F_{y2} - y_{lc}F_{x2} + T_{z1} + T_{z2} \\
& + \text{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w)\dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w)\dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c)\dot{\mathbf{q}}_w \end{pmatrix} mN_1 (y_c \sin \mathbf{q}_c - x_c \cos \mathbf{q}_c) \\
& + m \begin{pmatrix} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ - \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \frac{L}{2} + \mathbf{I}_2 (x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c)
\end{aligned} \tag{68}$$

$$M_{41}\ddot{x}_w + M_{42}\ddot{y}_w + M_{43}\ddot{\mathbf{q}}_w + M_{44}\ddot{\mathbf{q}}_c + C(4) = -\mathbf{I}_2 L_c - k\dot{\mathbf{q}}_c \tag{69}$$

Simplify (67), we have:

$$\begin{aligned}
& (M_{21} - tg\mathbf{q}_w M_{11})\ddot{x}_w + (M_{22} - tg\mathbf{q}_w M_{12})\ddot{y}_w + (M_{23} - tg\mathbf{q}_w M_{13})\ddot{\mathbf{q}}_w + (M_{24} - tg\mathbf{q}_w M_{14})\ddot{\mathbf{q}}_c + C(2) = \\
& F_{x1} \sin \mathbf{q}_w + F_{y1} \cos \mathbf{q}_w + F_{x2} \sin \mathbf{q}_w + F_{y2} \cos \mathbf{q}_w - \text{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w)\dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w)\dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c)\dot{\mathbf{q}}_w \end{pmatrix} mN_1 \cos(\mathbf{q}_c - \mathbf{q}_w) \\
& - m \begin{pmatrix} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \cos \mathbf{q}_w \\
& + \frac{\sin \mathbf{q}_w}{\cos \mathbf{q}_w} \left\{ \begin{array}{l} C(1) + \mathbf{I}_2 \cos(\mathbf{q}_c - \mathbf{q}_w) \\ - F_{x1} \cos \mathbf{q}_w + F_{y1} \sin \mathbf{q}_w - F_{x2} \cos \mathbf{q}_w + F_{y2} \sin \mathbf{q}_w \\ + \text{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w)\dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w)\dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c)\dot{\mathbf{q}}_w \end{pmatrix} mN_1 \sin(\mathbf{q}_c - \mathbf{q}_w) \\ - m \begin{pmatrix} \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \text{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \sin \mathbf{q}_w \end{array} \right\} + \mathbf{I}_2 \sin(\mathbf{q}_c - \mathbf{q}_w)
\end{aligned} \tag{70}$$

(70), (68), (69) and (55) together give:

$$SA_{11} = M_{21} - tg\mathbf{q}_w M_{11}$$

$$SA_{12} = M_{22} - tg\mathbf{q}_w M_{12}$$

$$SA_{13} = M_{23} - tg\mathbf{q}_w M_{13}$$

$$SA_{14} = M_{24} - tg\mathbf{q}_w M_{14}$$

$$\begin{aligned}
SA_{21} &= M_{31} \\
SA_{22} &= M_{32} \\
SA_{23} &= M_{33} \\
SA_{24} &= M_{34} \\
Sa_{31} &= M_{41} \\
Sa_{32} &= M_{42} \\
Sa_{33} &= M_{43} \\
Sa_{34} &= M_{44} \\
SA_{41} &= \cos \mathbf{q}_w \\
SA_{42} &= \sin \mathbf{q}_w \\
SA_{43} &= SA_{44} = 0
\end{aligned}
\tag{71}$$

$$\begin{aligned}
& \begin{matrix} \dot{x}_w; \\ \dot{y}_w; \\ \dot{\mathbf{q}}_w; \\ \dot{\mathbf{q}}_c; \\ -C(2) + F_{x1} \sin \mathbf{q}_w + F_{y1} \cos \mathbf{q}_w + F_{x2} \sin \mathbf{q}_w + F_{y2} \cos \mathbf{q}_w \\ - \operatorname{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{pmatrix} \mathbf{m} N_1 \cos(\mathbf{q}_c - \mathbf{q}_w) \\ - \mathbf{m} \begin{pmatrix} \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \cos \mathbf{q}_w \\ + \frac{\sin \mathbf{q}_w}{\cos \mathbf{q}_w} \left[\begin{matrix} C(1) + \mathbf{I}_2 \cos(\mathbf{q}_c - \mathbf{q}_w) \\ - F_{x1} \cos \mathbf{q}_w + F_{y1} \sin \mathbf{q}_w - F_{x2} \cos \mathbf{q}_w + F_{y2} \sin \mathbf{q}_w \\ + \operatorname{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{pmatrix} \mathbf{m} N_1 \sin(\mathbf{q}_c - \mathbf{q}_w) \\ - \mathbf{m} \begin{pmatrix} \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ + \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \sin \mathbf{q}_w \end{matrix} \right] + \mathbf{I}_2 \sin(\mathbf{q}_c - \mathbf{q}_w); \\ -C(3) - x_{lc} F_{y1} - y_{lc} F_{x1} + x_{lc} F_{y2} - y_{lc} F_{x2} + T_{z1} + T_{z2} \\ + \operatorname{sign} \begin{pmatrix} \sin(\mathbf{q}_c - \mathbf{q}_w) \dot{x}_w \\ + \cos(\mathbf{q}_c - \mathbf{q}_w) \dot{y}_w \\ + (x_c \cos \mathbf{q}_c - y_c \sin \mathbf{q}_c) \dot{\mathbf{q}}_w \end{pmatrix} \mathbf{m} N_1 (y_c \sin \mathbf{q}_c - x_c \cos \mathbf{q}_c) \\ + \mathbf{m} \begin{pmatrix} \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w - \frac{L}{2} \dot{\mathbf{q}}_w \right) N_2 \\ - \operatorname{sign} \left(-\sin \mathbf{q}_w \dot{x}_w + \cos \mathbf{q}_w \dot{y}_w + \frac{L}{2} \dot{\mathbf{q}}_w \right) N_3 \end{pmatrix} \frac{L}{2} + \mathbf{I}_2 (x_c \sin \mathbf{q}_c + y_c \cos \mathbf{q}_c + L_c); \\ -C(4) - \mathbf{I}_2 L_c - k \dot{\mathbf{q}}_c; \\ \sin \mathbf{q}_w \dot{\mathbf{q}}_w \dot{x}_w - \cos \mathbf{q}_w \dot{\mathbf{q}}_w \dot{y}_w \end{matrix} \end{matrix} \quad (72) \\
\dot{x} = A^{-1} B, \quad A^{-1} = \begin{bmatrix} [\mathbf{I}_{4 \times 4}] & [0]_{4 \times 4} \\ [0]_{4 \times 4} & [\mathbf{SA}]_{8 \times 8}^{-1} \end{bmatrix}
\end{aligned}$$

4. All nonholonomic constraints will not exist when $\mathbf{I}_1 > \mathbf{m}_n(N_2 + N_3)$ or $\mathbf{I}_1 < -\mathbf{m}_n(N_2 + N_3)$ and at the same time $\mathbf{I}_2 > \mathbf{m}_n N_1$ or $\mathbf{I}_2 < -\mathbf{m}_n N_1$, then take $\mathbf{I}_1 = \mathbf{m}_n(N_2 + N_3)$ or $\mathbf{I}_1 = -\mathbf{m}_n(N_2 + N_3)$ and $\mathbf{I}_2 = \mathbf{m}_n N_1$ or $\mathbf{I}_2 = -\mathbf{m}_n N_1$.

$$\dot{x} = A^{-1}B, \quad A^{-1} = \begin{bmatrix} [I]_{4 \times 4} & [0]_{4 \times 4} \\ [0]_{4 \times 4} & [M]^{-1}_{4 \times 4} \end{bmatrix}_{8 \times 8}, \quad B = Q - C$$

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