CS 4102, Algorithms: Recurrences, D & C

- First design strategy: Divide and Conquer
    - Examples…
  - Recursive algorithms
  - Counting basic operations in recursive algorithms: Solving recurrence relations
    - By iteration method
    - Recursion trees (quick view)
    - The “Main” and “Master” Theorems
- Mergesort
- Trominos

Recursion: Basic Concepts and Review

- Recursive definitions in mathematics
  - Factorial: n! = n (n-1)! and 0! = 1! = 1
  - Fibonacci numbers:

    F(0) = F(1) = 1

    F(n) = F(n-1) + F(n-2) for n > 1
  - Note base case
- In programming, recursive functions can be implemented
  - First, check for simple solutions and solve directly
  - Then, solve simpler subproblem(s) by calling same function
  - Must make progress towards base cases
- Design strategy: *method99* “mental trick”

Designing Recursive Procedures

- Think Inductively!
- converging to a base case (stopping the recursion)
  - identify some unit of measure (running variable)
  - identify base cases
- How to solve p for all inputs from size 0 through 100
  - Assume *method99* solves sub-problem all sizes 0 through 99
  - if p detect a case that is not base case it calls *method99*
- *method99* works and is called when:
  1. The sub-problem size is less than p's problem size
  2. The sub-problem size is not below the base case
  3. The sub-problem satisfies all other preconditions of *method99* (which are the same as the preconditions of p)

Recursion: Good or Evil?

- It depends…
- Sometimes recursion is an efficient design strategy, sometimes not
  - Important! we can define recursively and implement non-recursively
- Note that many recursive algorithms can be re-written non-recursively
  - Use an explicit stack
  - Remove tail-recursion (compilers often do this for you)
- Consider: factorial, binary search, Fibonacci
  - Let’s consider Fibonacci carefully…

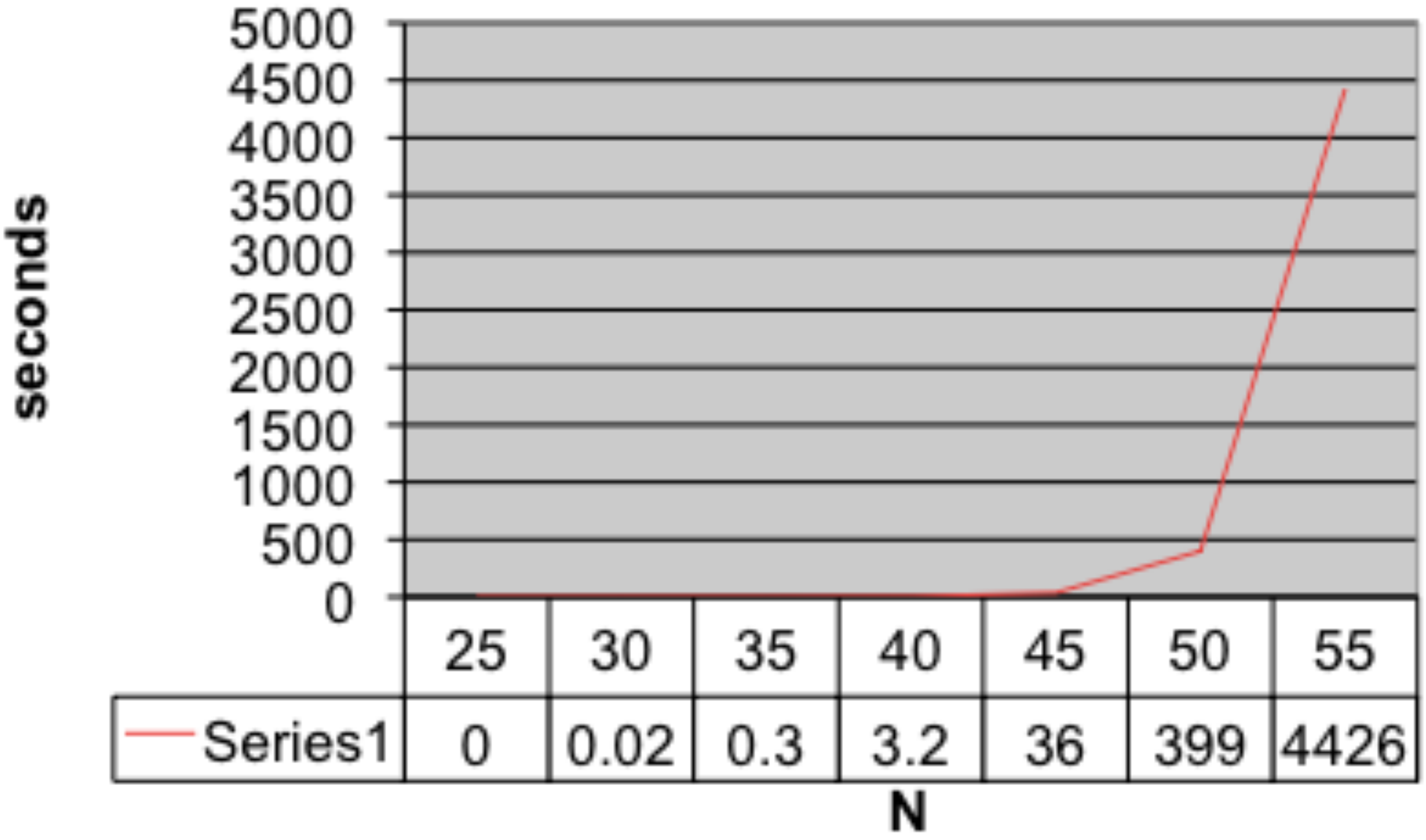
Implement Fibonacci numbers

- It’s beautiful code, no?

```
long fib(int n) {
    assert(n >= 0);
    if ( n == 0 ) return 1;
    if ( n == 1 ) return 1;
    return fib(n-1) + fib(n-2);
}
```

- Let’s run and time it.
- Let’s trace it.

Time for Recursive Fibonacci



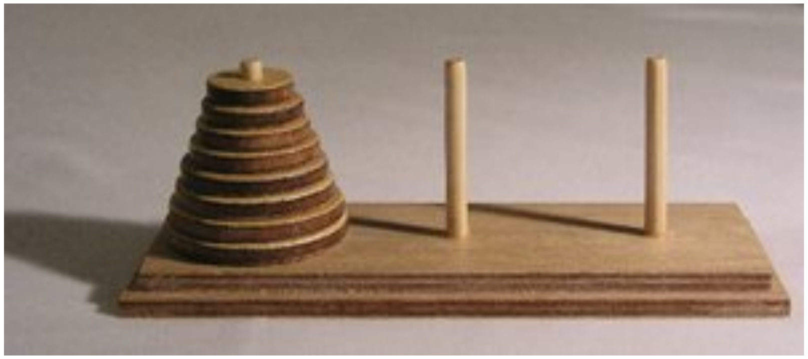
N	25	30	35	40	45	50	55
Series1	0	0.02	0.3	3.2	36	399	4426

Towers of Hanoi

- Ah, the legend:
  - 64 golden disks
  - Those diligent priests
  - The world ends!

Towers of Hanoi

- Back in the commercial Western world…
- Game invented by the French mathematician, Edouard Lucas, in 1883.
- Now, for only $19.95, call now!



Wake Up and Design!

- Write a recursive function for the Towers of Hanoi.
  - Number each peg: 1, 2, 3
  - Function signature:
    hanoi (n, source, dest, aux)
    where:
    n is number of disks (from the top), and
    other parameters are peg values
    In function body print:
    Move a disk from <peg> to <peg>
- Do this in pairs. Then pairs group and compare. Find bugs, issues, etc. Explain to each other. Turn in one sheet with all four names.

Divide and Conquer: A Strategy

- Our first design strategy: Divide and Conquer
- Often recursive, at least in definition
- Strategy:
  - Break a problem into 1 or more smaller subproblems that are identical in nature to the original problem
  - Solve these subproblems (recursively)
  - Combine the results for the subproblems (somehow) to produce a solution to original problem
- Note the assumption:
  - We can solve original problem given subproblems’ solutions

Design Strategy: Divide and Conquer

- It is often easier to solve several small instances of a problem than one large one.
  - **divide** the problem into smaller instances of the same problem
  - solve (**conquer**) the smaller instances recursively
  - **combine** the solutions to obtain the solution for original input
  - Must be able to solve one or more small inputs **directly**
- Solve(I)

```
n = size(I)
if (n <= smallsize)
    solution = directlySolve(I);
else
    divide I into I1, …, Ik.
    for each i in {1, …, k}
        Si = solve(Ii);
    solution = combine(S1, …, Sk);
return solution;
```

Why Divide and Conquer?

- Sometimes it’s the simplest approach
- Divide and Conquer is often more efficient than “obvious” approaches
  - E.g. Mergesort, Quicksort
- But, not necessarily efficient
  - Might be the same or worse than another approach

- Must analyze cost

- Note: divide and conquer may or may not be implemented recursively

Cost for a Divide and Conquer Algorithm

- Perhaps there is…
  - A cost for dividing into sub problems
  - A cost for solving each of several subproblems
  - A cost to combine results
- So (for *n > smallSize*)

  $$T(n) = D(n) + \Sigma\, T(size(I_i)) + C(n)$$

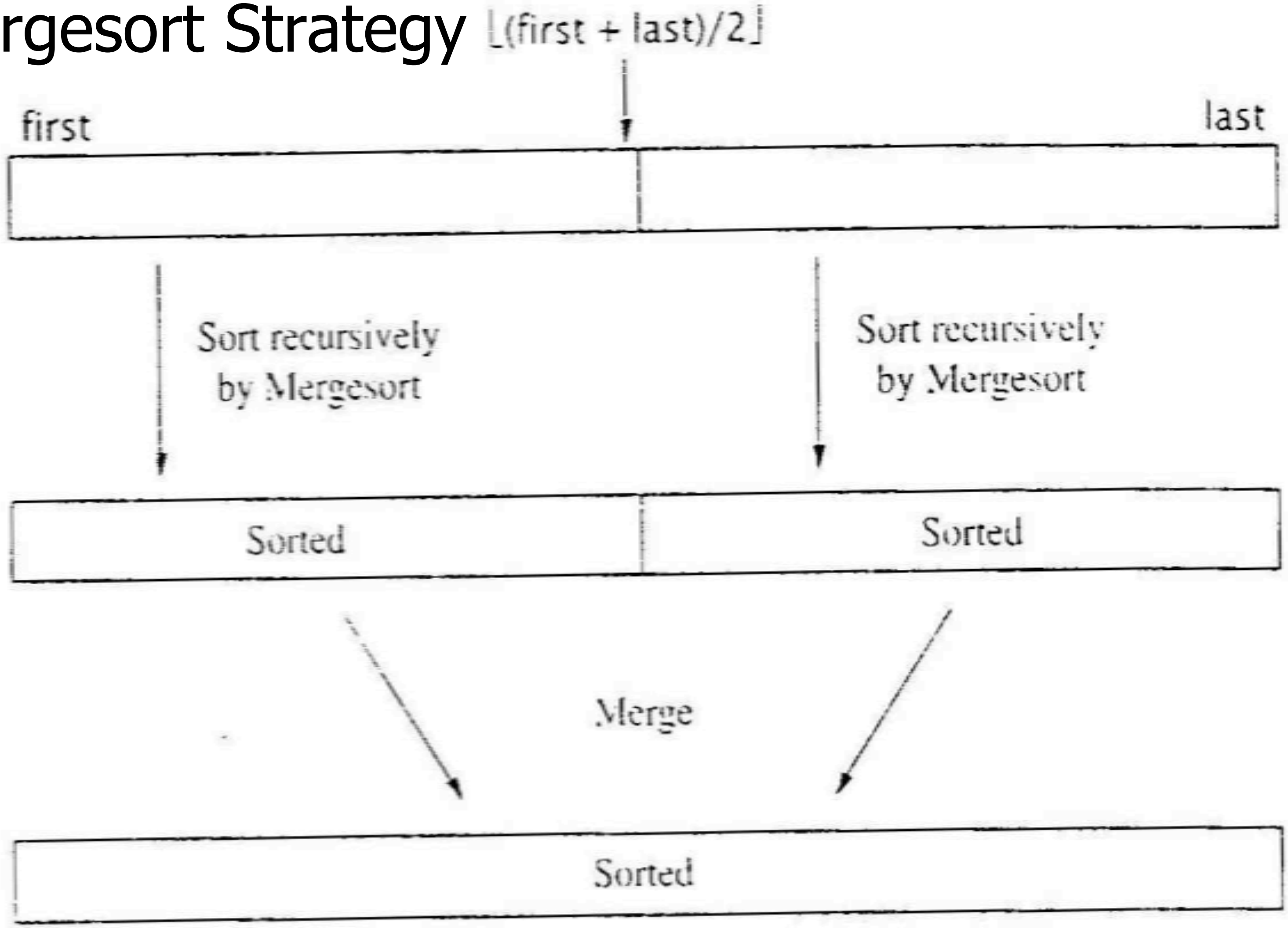
  - often rewritten as

  $$T(n) = a\, T(n/b) + f(n)$$

- These formulas are *recurrence relations*

Mergesort is Classic Divide & Conquer

- Mergesort Strategy



Algorithm: Mergesort

- Specification:
  - Input: Array E and indexes first, and Last, such that the elements E[i] are defined for first <= i <= last.
  - Output: E[first], …, E[last] is sorted rearrangement of the same elements
- Algorithm:

```
def mergesort(list, first, last):
    if first < last:
        mid = (first+last)/2
        mergesort(list, first, mid)
        mergesort(list, mid+1, last)
        merge(list, first, mid, last) # merge 2 halves
    return
```

Exercise: Find Max and Min

- Given a list of elements, find both the maximum element and the minimum element
- Obvious solution:
  - Consider first element to be max
  - Consider first element to be min
  - Scan linearly from 2nd to last, and update if something larger then max or if something smaller than min
- Class exercise:
  - Write a recursive function that solves this using divide and conquer.
    - Prototype: void maxmin (list, first, last, max, min);
    - Base case(s)? Subproblems? How to combine results?

Solving Recurrence Relations

- Several methods:
  - Substitution method, AKA iteration method, AKA method of backwards substitutions
    - We'll do this in class
  - Recurrence trees
    - Not in our text. (In the Baase text from 2003.)
    - Sometimes a picture is worth 2^{10} words!
  - "Main" Theorem and the "Master" theorem
    - Easy to find Order-Class for a number of common cases
    - Textbook: Main Theorem
    - Other texts: slightly different Master Theorem

Iteration or Substitution Method

- Strategy
  - Write out recurrence, e.g. $W(n) = W(n/2) + 1$
    - BTW, this is a recurrence for binary search
  - Substitute for the recursive definition on the right-hand side by re-applying the general formula with the smaller value
    - In other words, plug the smaller value back into the main recurrence
  - So now: $W(n) = (W(n/4) + 1) + 1$
  - Repeat this several times and write it in a general form (perhaps using some index i to show how often it's repeated)
  - So now: $W(n) = W(n/2^i) + i$

Substitution Method (cont'd)

- So far we have: $W(n) = W(n/2^i) + i$
- This is the form after we repeat i times. How many times can we repeat?
  - Use base case to solve for i
  - Here, $W(1) = 1$, so we reach this when $n/2^i$ is 1.
    - Solve for i: so $i = \lg n$
  - Plug this value of i back into the general recurrence:

    $$W(n) = W(n/2^i) + i = W(n/n) + \lg n = \lg n + 1$$
  - Note: We assume n is some power of 2, right?
    - That's OK. There is a theorem called the smoothness rule that states that we'll have the correct order-class
    - See Example 2.4.6, page 58

Examples Using the Substitution Method

Practice with the following:

1. Finding max and min

   $W(1) = 0, \quad W(n) = 2\,W(n/2) + 2$

   - Is this better or worse than the "scanning" approach?

2. Mergesort

   $W(1) = 0, W(n) = 2\,W(n/2) + n - 1$

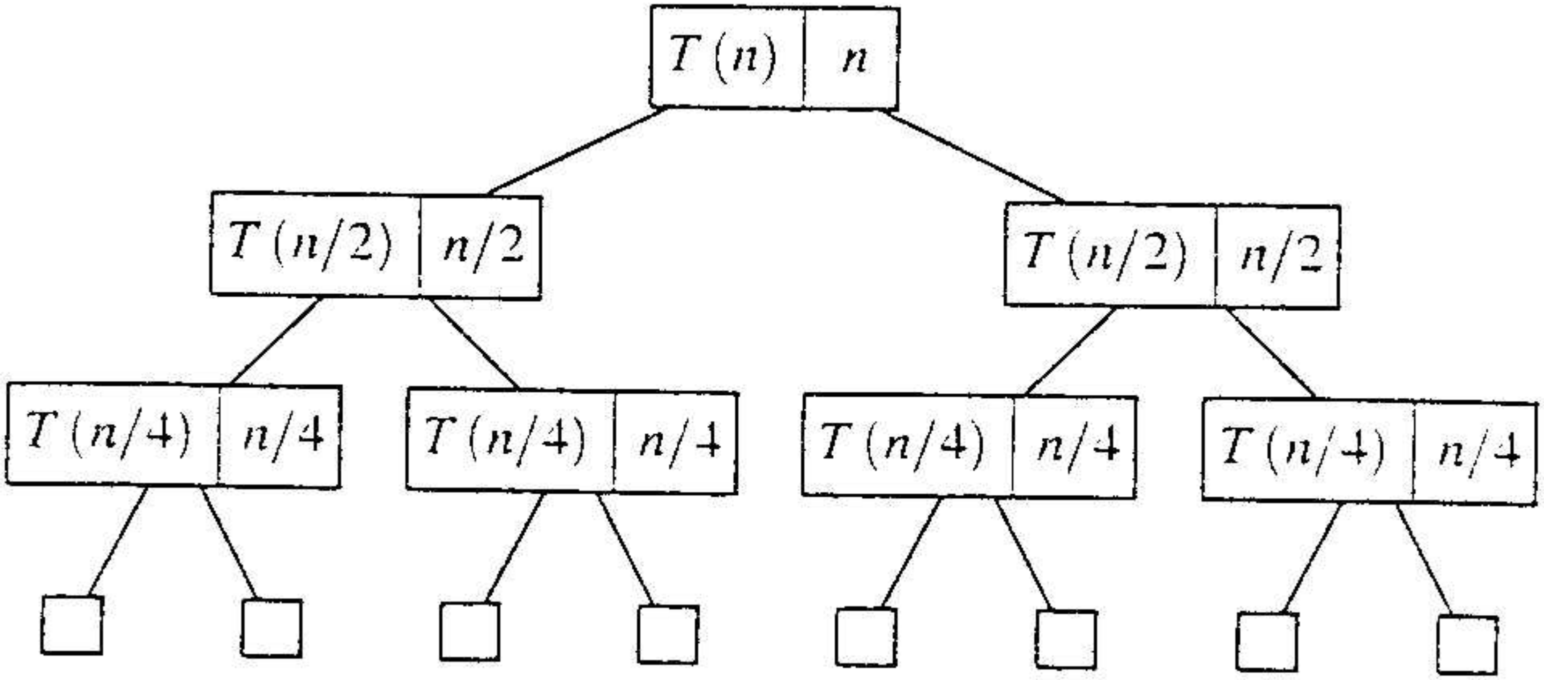
3. Towers of Hanoi
   - Write the recurrence. (Now, in class.)
   - Solve it. (At home!)

Return to Fibonacci…

- Can we use the substitution method to find out the W(n) for our recursive implementation of fib (n)?
  - Nope. There's another way to solve recurrence, which we won't do in this class
    - homogenous second-order linear recurrence with constant coefficients
  - This method allows us to calculate F(n) "directly":
    $F(n) = (1 / \text{sqrt}(5))\ \Phi^n$ rounded to nearest int,
    where Φ is the Golden Ratio, about 1.618
  - Isn't this $\Theta(1)$ while a loop is $\Theta(n)$? (Just punch buttons on my calculator!)
    - Without a table or a calculator, finding Φ^n is linear (just like finding F(n) with a loop)

Evaluate recursive equation using Recursion Tree

- Evaluate: T(n) = T(n/2) + T(n/2) + n
    - Work copy: T(k) = T(k/2) + T(k/2) + k
    - For k=n/2, T(n/2) = T(n/4) + T(n/4) + (n/2)
- [size| non-recursive cost]



Recursion Tree: Total Cost

- To evaluate the total cost of the recursion tree
  - sum all the non-recursive costs of all nodes
  - = Sum (rowSum(cost of all nodes at the same depth))
- Determine the maximum depth of the recursion tree:
  - For our example, at tree depth d
    the size parameter is $n/(2^d)$
  - the size parameter converging to base case, i.e. case 1
  - such that, $n/(2^d) = 1$,
  - $d = \lg(n)$
  - The rowSum for each row is n
- Therefore, the total cost, $T(n) = n \lg(n)$

The Master Theorem

- Given: a *divide and conquer* algorithm
  - An algorithm that divides the problem of size *n* into *a* subproblems, each of size *n*/b
  - Let the cost of each stage (i.e., the work to divide the problem + combine solved subproblems) be described by the function *f(n)*
- Then, the Master Theorem gives us a cookbook for the algorithm's running time
  - Our textbook has a simpler version they call the "Main Recurrence Theorem"

The Master Theorem (from Cormen's)

- If $T(n) = aT(n/b) + f(n)$ then
  let $k = \lg a / \lg b = \log_b a$ (critical exponent)
- Then three common cases based on how quickly $f(n)$ grows
  1. If $f(n) \in O(n^{k-\varepsilon})$ for some positive ε,
     then $T(n) \in \Theta(n^k)$
  2. If $f(n) \in \Theta(n^k)$
     then $T(n) \in \Theta(\ f(n) \log(n)\ ) = \Theta(n^k \log(n)\ )$
  3. If $f(n) \in \Omega(n^{k+\varepsilon})$ for some positive ε, and
     $f(n) \in O(n^{k+\delta})$ for some positive $\delta >= \varepsilon$,
     then $T(n) \in \Theta(f(n))$
- Note: none of these cases may apply

The Main Recurrence Theorem (from our text)

- If $T(n) = aT(n/b) + f(n)$ and
  $f(n) = \Theta(n^k)$
- Cases for exact bound:
  1. $T(n) \in \Theta(n^k)$ if $a < b^k$
  2. $T(n) \in \Theta(\ n^k \log(n)\ )$ if $a = b^k$
  3. $T(n) \in \Theta(n^E)$ where $E=\log_b(a)$ if $a > b^k$
- Note book's similar cases for upper and lower bound
- Note $f(n)$ is polynomial
  - This is less general than earlier Master Theorem

Using The Master Method

- $T(n) = 9T(n/3) + n$
  - $a=9, b=3, f(n) = n$
- Main Recurrence Theorem
  - $a ? b^k$ $9 > 3$, so $\Theta(n^E)$ where $E=\log_3(9) = 2$, $\Theta(n^2)$
- Master Theorem
  - $k = \lg 9 / \lg 3 = \log_3 9 = 2$
  - Since $f(n) = O(n^{\log_3 9 - \varepsilon})$, where $\varepsilon=1$, case 1 applies:

$$T(n) \in \Theta(n^E)$$

  - Thus the solution is $T(n) = \Theta(n^2)$ since $E=2$

Problems to Try

- Can you use a theorem on these?
  Can you successfully use the iteration method?

- $T(n) = T(n/2) + \lg n$
- $T(n) = T(n/2) + n$
- $T(n) = 2T(n/2) + n$ (like Mergesort)
- $T(n) = 2T(n/2) + n \lg n$

Common Forms of Recurrence Equations

- Recognize these:
  - Divide and conquer
    $T(n) = bT(n/c) + f(n)$
    - Solve directly or apply master theorem
  - Chip and conquer:
    $T(n) = T(n-c) + f(n)$
    - Note: One subproblem of lesser cost!
  - Chip and <u>Be</u> Conquered:
    $T(n) = b\,T(n-c) + f(n)$ where $b > 1$
    - Like Towers of Hanoi

Back to Towers of Hanoi

- Recurrence:

  $W(1) = 1; \qquad W(n) = 2\,W(n-1) + 1$

- Closed form solution:

  $W(n) = 2^n - 1$

- Original "legend" says the monks moves 64 golden disks
  - And then the world ends! (Uh oh.)
  - That's 18,446,744,073,709,551,615 moves!
  - If one move per second, day and night, then 580 billion years
  - Whew, that's a relief!