CS 4102: Algorithms
NP Completeness

Chapter 10 in Johnsonbaugh & Schaefer

Read and study! (This ain't simple.)

Slide credits: Thanks to David Luebke, Jim Cohoon

NP-Completeness

- Some problems are *intractable*:
  as they grow large, we are unable to solve them in reasonable time
- What constitutes reasonable time? Standard working definition: *polynomial time*
  - On an input of size n the worst-case running time is $O(n^k)$ for some constant k
  - Polynomial time: $O(n^2)$, $O(n^3)$, $O(1)$, $O(n \lg n)$
  - Not in polynomial time: $O(2^n)$, $O(n^n)$, $O(n!)$

Polynomial-Time Algorithms

- Are some problems solvable in polynomial time?
  - Of course: many (most?) algorithms we've studied provides polynomial-time solution to some problem
  - We define **P** to be the class of problems solvable in polynomial time
- Are all problems solvable in polynomial time?

Tractability

- Again, some problems are *undecidable*: no computer can solve them
  - E.g., Turing's "Halting Problem"
  - We're not going to talk about such problems here (Take a theory class to learn more!)
- Other problems are decidable, but *intractable*: as they grow large, we are unable to solve them in reasonable time
  - *What constitutes "reasonable time"?*

Flashback: Growth Rates

- Review pages 430-431 for this important point:
  - Say *s* is the size of the largest problem we can solve with algorithm *A* given time *t*
  - How much larger a problem can we solve if we have 10x as much time (or a computer that's 10x faster)?
- Depends on the complexity of *A*
  - Linear? 10 x s times as large a problem
  - Quadratic? sqrt(10) = 3.2 x s times as large
  - Cubic? cuberoot(10) = 2.2 x s times as large
  - 2^n?
    - s + lg 10 as large, or s + 3.2 as large
    - Note additive factor, not mulitplicative

NP-Complete Problems

- The *NP-Complete* problems are an interesting class of problems whose status is unknown
  - No polynomial-time algorithm has been discovered for an NP-Complete problem
  - No suprapolynomial lower bound has been proved for any NP-Complete problem, either
- We call this the *P = NP question*
  - The biggest open problem in CS

An NP-Complete Problem: Hamiltonian Cycles

- An example of an NP-Complete problem:
  - A *hamiltonian cycle* of an undirected graph is a simple cycle that contains every vertex
  - The hamiltonian-cycle problem: given a graph G, does it have a hamiltonian cycle?

  - *Describe a naïve algorithm for solving the hamiltonian-cycle problem. Running time?*
  - *Have we studied a search algorithm that can be used to solve this? Running time?*

Other Problems to Know

- Hamilton Path/Cycle, TSP
- Graph Coloring
- Vertex cover
- Satisifiability
- Subset Sum
- Bin Packing
- Knapsack

Some Definitions Before We Proceed

- Decision problems
  - Simple view: Problem is to answer with a "yes" or "no" answer for a given input
- Definition, page 431:
  - **A problem** is a set of finite-length questions (strings) with associated finite-length answers (strings).
  - **A decision problem:** all questions (instances) map to either yes or no
    - Positive instances vs. negative instances
  - A correct algorithm ***accepts*** all positive instances (says yes) and ***rejects*** negative instances (says no)

Decision Problems and Related Problems

- Problem P1: Primality Problem
  - Is n a prime number?
  - Instances: set of natural numbers
    - Positive instances?
- Related Problem P2: Find smallest divisor
  - Not a decision problem.
  - P2 could be used to solve P1.
  - In some sense, P2 is "harder".
- Graph coloring
  - P3: k-colorability. Given k, this is a decision problem
  - P4: Find chromatic number? not a decision problem
    - Could use k-colorability to solve it.

Reminder: Graph Coloring

- A “coloring” is an assignment of “colors” to the set of vertices so that if *vw* is an edge, then C(v) <> C(w)
- Two forms:
  - Optimization problem: given *G*, find the smallest number of colors that can be used to color G
    - The smallest k is known as G’s *chromatic number, X* (G) We say G is k-colorable.
  - Decision problem: given *G* and a positive integer *k*, is it possible to assign a valid k-coloring to *G*?
  - Note: book uses term *function problem*
    - More general. An optimization is a type of…

Decision vs. Optimization

- Clearly an optimization problem is related to a particular decision problem
  - Decision problem: Is there a solution as good or better than some given bound?
  - Optimal value: What is the value of the best possible solution?
  - Optimal solution: Find a solution with the optimal value!
- E.g. graph coloring
  - Decision problem: Can G be colored with k colors?
  - Optimal value: What is the chromatic number of G?
  - Optimal solution: Find a coloring of vertices that uses X (G) colors.

Optimization and Decision Problems

- Optimization problems are at least as hard to solve as decision problems
  - E.g. if you can solve decision problem canColor(G,k) then call it in a loop (1 to n) to find optimal value, X(G)

- Important: theory presented here for P, NP etc. is defined based on <u>decision problems</u>
  - But we'll see this isn't a problem…

Encodings, Input Sizes

- Our text takes a formal CS approach to these topics
  - Languages, accepting, encodings, etc.
- I choose in CS4102 to be less formal
  - So I'll try to "translate" or simplify when I can

- So about pages 433-434 on encodings…

Important: Input Size and P

- Sometimes a problems seems to be in P but really isn't
- Example: finding if value n is a prime
  - Just loop and do a mod: $\Theta(n)$, isn't it?
- Note that here "n" is not the count or number of data items.
  - There's just one input item.
  - But "n" is a value with a <u>size</u> that affects the execution time.
  - The size is the number of bits, which is log(n)
  - T(size) = n but size is log(n).
  - $T(\log n) = n = 10^{\log n}$ This is really an exponential!
- Be careful when "n" is not a count of data items but a value
  - E.g. Dynamic programming problems (e.g. a table's dimension)

P and NP

- As mentioned, **P** is set of decision problems that can be solved in polynomial time
- **NP** (*nondeterministic polynomial time*) is the set of decision problems that can be solved in polynomial time by a *nondeterministic* computer
  - *What on earth is that?*
  - *Important: "NP" does not mean "not polynomial"!!!*

Nondeterminism

- Think of a non-deterministic computer as a computer that magically "guesses" a solution, then has to verify that it is correct
  - If a solution exists, computer always guesses it
  - One way to imagine it: a parallel computer that can freely spawn an infinite number of processes
    - Have one processor work on each possible solution
    - All processors attempt to verify that their solution works
    - If a processor finds it has a working solution
  - So: **NP** = problems *verifiable* in polynomial time

Nondeterminism (cont'd)

- Another way to think about it:
  - If you could always guess the answer but were required to verify your guess was really right, could you do this in polynomial time?
    - an *oracle*
- Just a second: solutions for decision problems are "yes" or "no"
  - How can guessing that help us?
  - The idea of verifying a solution in polynomial time is an informal definition of nondeterminism.

Nondeterminism and our Text

- Pages 440-441
  - A *guess* function that makes a choice
  - Count it as one step (i.e. constant complexity)
  - If run again, could guess something else
    - Non-deterministic
  - Sequence of guesses is a *computation path* or *run*
  - If some run leads to a “yes”, then that sequence of choices is a *witness*
    - It proves that input is accepted by the algorithm

Nondeterminism More Formally

- Non-deterministic algorithm **A** has 2 phases:
  - Non-deterministic phase: writes a string *s* (often called a *certificate*) somewhere
  - Deterministic phase: May use *s* and the input to return "yes" or "no" or nothing
- When run with same input, may produce different certificate *s*
- Total cost is cost of both steps
- Important: If A is give input x and for <u>*some*</u> execution it says "yes", then the answer is "yes".

How to Think about Non-Det. Here

- Two ways:
  - If we ran a non-deterministic algorithm once and it made the right guess each time (wrote the correct certificate) if that was possible…
  - Or, if we ran it so many times over all possibilities, and one of those led to a "yes"…
- If some oracle could write out the "right" certificate, would we recognize it and say "yes"?
- (An aside: If not, then we're not able to check a correct answer. That's not good, is it?)

Definition of Class NP

- Definition:
  NP is the class of decision problems for which there is a polynomially bounded non-deterministic algorithm.
- Reminder: formally these are decision problems
  - But (for many problems) imagine a certificate that is the optimal solution
  - E.g. 3-colorability: *s* is a coloring
  - E.g. ham. path: *s* is a path
  - Can we "verify" these and say "yes" in poly. time?

Proving Problems are in NP

- Self test:
  - *How do we typically prove a problem $\in$ **NP**?*
- Examples:
  - *Is sorting in **NP**? (Not a decision problem! Could redefine it to be.)*
    - What could the certificate be? Could we verify it in poly. time?
  - Is this in **NP**? Does a weighted graph G have a spanning tree with value <= k?
    - Think-Pair-Share activity: prove this belongs to NP
      - What's the certificate? What do the two phases do?

Proving Problems are in NP (2)

- Note: The non-deterministic phase might do "nothing". Example:
- Problem: Does a graph G have a MST of total weight less than k?
  - Does this belong to NP? Yes! Outline of proof:
    - No need to generate a certificate or guess non-deterministically
    - Find MST using Kruskal's or Prim's algorithms
    - Compare weight of the MST to k
    - Thus we can verify a proposed yes/no answer in polynomial time
- Thus problems in P are easily shown to be in NP

P and NP

- *Is* ***P*** $\subseteq$ ***NP****? Why or why not?*
- Answer: all decision problems in P also belong to NP
  - Informally: you can solve them directly and compare the solution to the certificate
- But are they equal or is it a proper subset?
- In other words, is there a problem in **NP** that cannot be directly solved in polynomial time?
  - Is P = NP? Or not? (The big question!)

Summary: P and NP

- Summary so far:
  - **P** = problems that can be solved in polynomial time
  - **NP** = problems for which a solution can be verified in polynomial time
  - Unknown whether **P = NP** (most suspect not)
- We've seen problems that belong to **NP** that may not belong to P
  - Hamiltonian path/cycle, k-COL problems are in **NP**
  - Cannot solve in polynomial time
  - Easy to verify solution in polynomial time (*How?*)

NP-Complete Problems

- We will see that NP-Complete problems are the "hardest" problems in NP:
  - If any *one* NP-Complete problem can be solved in polynomial time…
  - …then *every* NP-Complete problem can be solved in polynomial time…
  - …and in fact *every* problem in **NP** can be solved in polynomial time (which would show **P = NP**)
  - Thus: solve hamiltonian-cycle in O(n^{100}) time, you've proved that **P = NP**. Retire rich & famous.

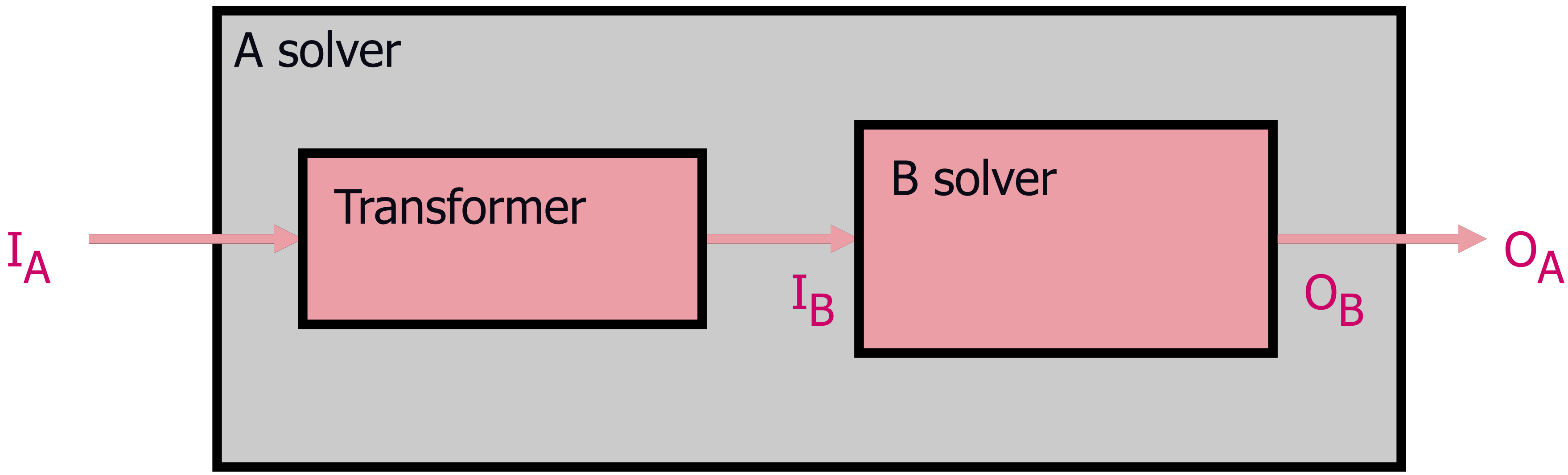
Reduction

- The crux of NP-Completeness is *reducibility*
  - Informally, a problem A can be reduced to another problem B if *any* instance of A can be "easily rephrased" as an instance of B , the solution to which provides a solution to the instance of A
    - *What do you suppose "easily" means?*
    - This rephrasing is called *transformation*
  - Intuitively: If A reduces to B , A is "no harder to solve" than B
    - Total cost: cost of transformation + cost to solve B

Reducibility

- An example:
  - A: Given a set of Booleans, is at least one TRUE?
  - B: Given a set of integers, is their sum positive?
  - Transformation: $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ where $y_i = 1$ if x_i = TRUE, $y_i = 0$ if x_i = FALSE
- Another example:
  - Solving linear equations is reducible to solving quadratic equations
    - *How can we easily use a quadratic-equation solver to solve linear equations?*

Reduction: A reduces to B



NP-Hard and NP-Complete

- If A is *polynomial-time reducible* to B, we denote this $A \leq_p B$
- Definition of NP-Hard and NP-Complete:
  - If all problems $X \in$ **NP** are reducible to A, then A is *NP-Hard*
  - We say A is *NP-Complete* if A is NP-Hard and $A \in$ **NP**
- If $A \leq_p B$ and A is NP-Complete, B is also NP-Complete
  - *You should be able to argue this is true from the definitions!*

Why Prove NP-Completeness?

- Though nobody has proven that **P ≠ NP**, if you prove a problem NP-Complete, most people accept that it is probably intractable
- Therefore it can be important to prove that a problem is NP-Complete
  - Don't need to come up with an efficient algorithm
  - Can instead work on *approximation algorithms*

Proving NP-Completeness

- *What steps do we have to take to prove a problem A is NP-Complete?*
  - Pick a known NP-Complete problem B
  - Reduce B to A
    - Describe a transformation that maps instances of B to instances of A, s.t. "yes" for A = "yes" for B
    - Prove the transformation works
    - Prove it runs in polynomial time
  - Oh yeah, prove A $\in$ **NP** (*What if you can't?*)

Proving NP-Completeness

- *(We just said this:) What steps do we have to take to prove a problem A is NP-Complete?*
  - Pick a known NP-Complete problem B
  - Reduce B to A

- Why reduce B to A? Transformations are transitive
  - If B is NP-c, then all problems in NP reduce to B.
  - Then, if you find a transformation from B to A, the *composition* of two transformations would reduce any NP problem to A
  - The composition of two polynomials is polynomial

Coming Up

- Given one NP-Complete problem, we can prove many interesting problems NP-Complete
  - Graph coloring (= register allocation)
  - Hamiltonian cycle
  - Hamiltonian path
  - Knapsack problem
  - Traveling salesman
  - Job scheduling with penalities
  - Many, many more