# CS 4102: Algorithms

NP Completeness Continued:
Reductions

# Review: **P** And **NP** Summary

- **P** = set of problems that can be solved in polynomial time
- **NP** = set of problems for which a solution can be verified in polynomial time
- **P** $\subseteq$ **NP**
- Open question: Does **P = NP**?

# Review: Reduction

- A problem A can be *reduced* to another problem B if any instance of A can be rephrased to an instance of B, the solution to which provides a solution to the instance of A
  - This rephrasing is called a *transformation*
- Intuitively: If A reduces in polynomial time to B, A is "no harder to solve" than B
  - I.e. if B is polynomial, A is not exponential

# Review: NP-Hard and NP-Complete

- If A is *polynomial-time reducible* to B, we denote this $A \leq_p B$
- Definition of NP-Hard and NP-Complete:
  - If all problems $R \in$ **NP** are reducible to A , then A is *NP-Hard*
  - We say A is *NP-Complete* if A is NP-Hard and $A \in$ **NP**
- If $A \leq_p B$ and A is NP-Complete, B is also NP- Complete

# Review: Proving NP-Completeness

- *What steps do we have to take to prove a problem Y is NP-Complete?*
  - Pick a known NP-Complete problem X
    - Assuming there is one! (More later.)
  - Reduce X to Y
    - Describe a transformation that maps instances of X to instances of Y, s.t. "yes" for Y = "yes" for X
    - Prove the transformation works
    - Prove it runs in polynomial time
  - Oh yeah, prove Y $\in$ **NP**

# Order of the Reduction When Proving NP-Completeness

- To prove Y is **NP-c**, show $X \le_p Y$ where $X \in$ **NP-c**
  - Why have the known NP-c problem "on the left"? Shouldn't it be the other way around? (No!)
- If $X \in$ **NP-c**, then: all NP problems $\le_p X$
- If you show $X \le_p Y$, then:

  any-NP-problem $\le_p X \le_p Y$

- Thus any problem in NP can be reduced to Y if the two transformations are applied in sequence
  - And both are polynomial

# Can a Problem be NP-Hard but not NP-C?

- So, find a reduction and then try to prove Y ∈ **NP**
  - *What if you can't?*
- Are there any problems Y that are NP-hard but not NP-complete? This means:
  - All problems in NP reduce to Y . (A known NP-c problem can be reduced to Q.)
  - But, Y cannot be proved to be in NP
- Yes! Some examples:
  - Non-decision forms of known NP-Cs (e.g. TSP)
  - The halting problem. (Transform a SAT expression to a Turing machine.)
  - Others.

# But You Need One NP-c First…

- If you have one NP-c problem, you can use the technique just described to prove other problems are NP-c
- The definition of NP-complete was created to prove a point
  - There *might be* problems that are at least as hard as "anything" (i.e. all NP problems)
- Are there really NP-complete problems?
  - Stephen Cook, 1971. **Cook-Levin Theorem: The satisfiability problem is NP-Complete.**
    - He proved this "directly", from first principles
    - Proven independently by Leonid Levin (USSR)
    - Showed that any problem that meets the definition of NP can be transformed in polynomial time to a CNF formula.
    - Proof outside the scope of this course (lucky you)

# More About The SAT Problem

- One of the first problems to be proved NP-Complete was *satisfiability* (SAT):
  - Given a Boolean expression on *n* variables, can we assign values such that the expression is TRUE?
  - Ex: $((x_1 \rightarrow x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$

- You might imagine that lots of decision problems could be expressed as a complex logical expression
  - And Cook and Levin proved you were right!
  - Proved the general result that any NP problem can be expressed

# Conjunctive Normal Form

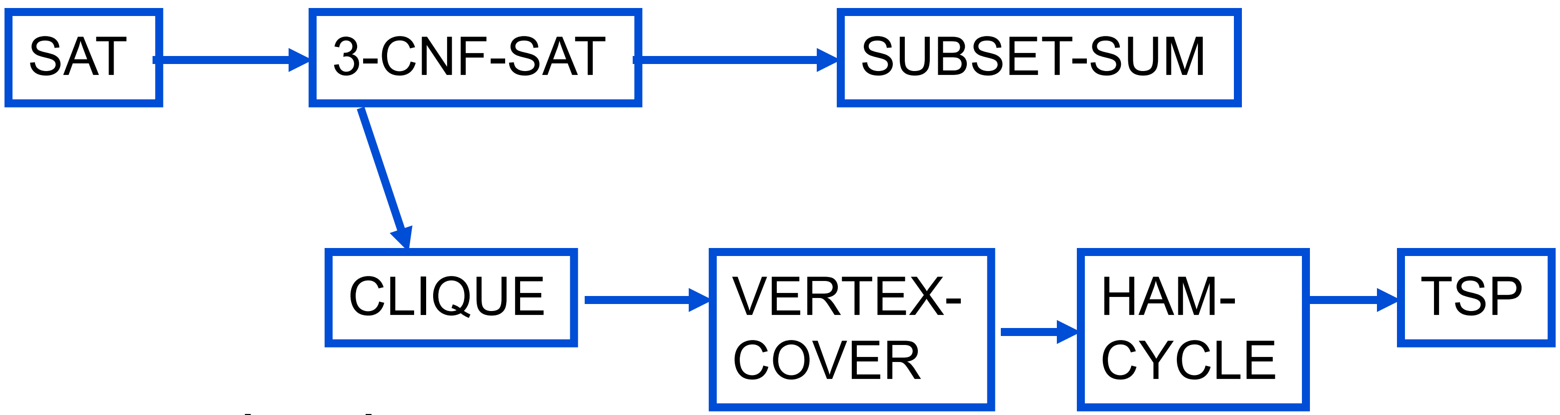
- Even if the form of the Boolean expression is simplified, the problem may be NP-Complete
  - *Literal*: an occurrence of a Boolean or its negation
  - A Boolean formula is in *conjunctive normal form*, or *CNF*, if it is an AND of clauses, each of which is an OR of literals
    - Ex: $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (\neg x_5)$
  - *3-CNF*: each clause has exactly 3 distinct literals
    - Ex: $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (\neg x_5 \vee x_3 \vee x_4)$
    - Notice: true if at least one literal in each clause is true
  - Note: Arbitrary expressions can be translated into CNF forms by introducing intermediate variables etc.

# The 3-CNF Problem

- Satisfiability of Boolean formulas in 3-CNF form (the *3-CNF Problem*) is NP-Complete
  - Proof: not in this course
- The reason we care about the 3-CNF problem is that it is relatively easy to reduce to others
  - Thus by proving 3-CNF NP-Complete we can prove many seemingly unrelated problems NP-Complete

# Joining the Club

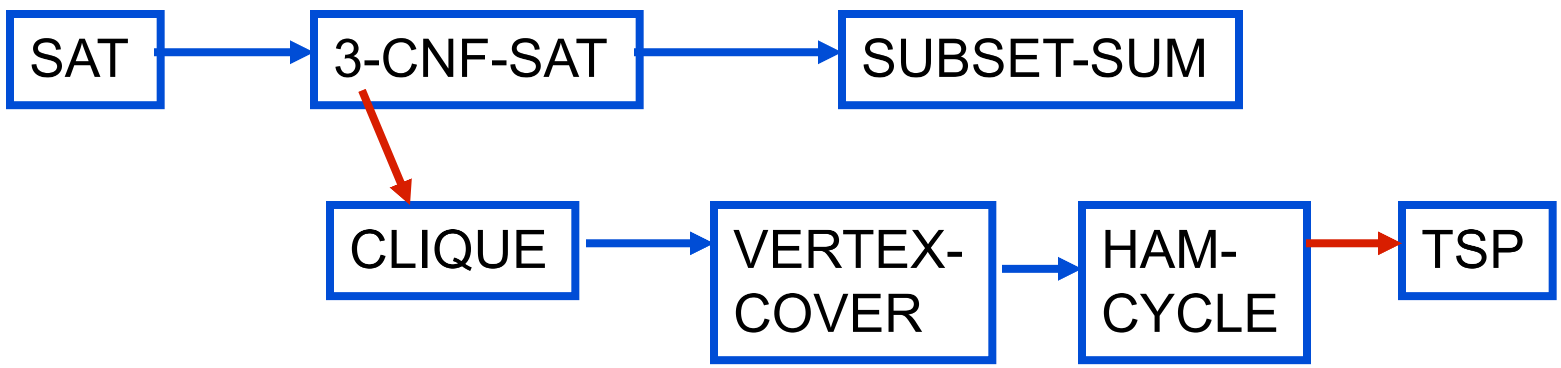
- Given one NP-c problem, others can join the club
  - Prove that SAT reduces to another problem, and so on…



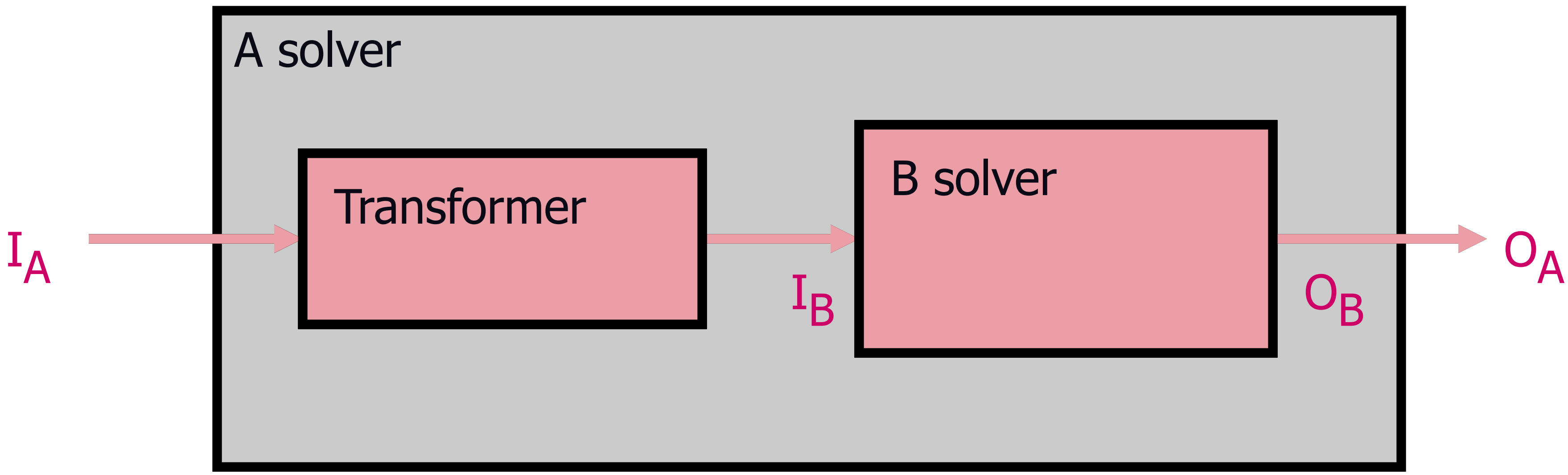

  - Membership in NP-c grows…
  - Classic textbook: Garey, M. and D. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness,* 1979.

# Examples of Reductions

- Examples covered in class:
  - 3-CNF to k-Clique (in these slides)
  - Directed to Undirected Hamilton Cycle (slides & handout)
  - Hamilton Cycle to Traveling Salesperson (in these slides)
  - 3-COL to CNF-SAT (handout shows direct reduction)



# Reminder: A reduces to B



# 3-CNF → Clique

- *What is a clique of a graph G?*
- A: a subset of vertices fully connected to each other, i.e. a complete subgraph of G
- The *clique problem*: how large is the maximum-size clique in a graph?
- *Can we turn this into a decision problem?*
- A: Yes, we call this the *k-clique problem*
- *Is the k-clique problem within **NP**?*

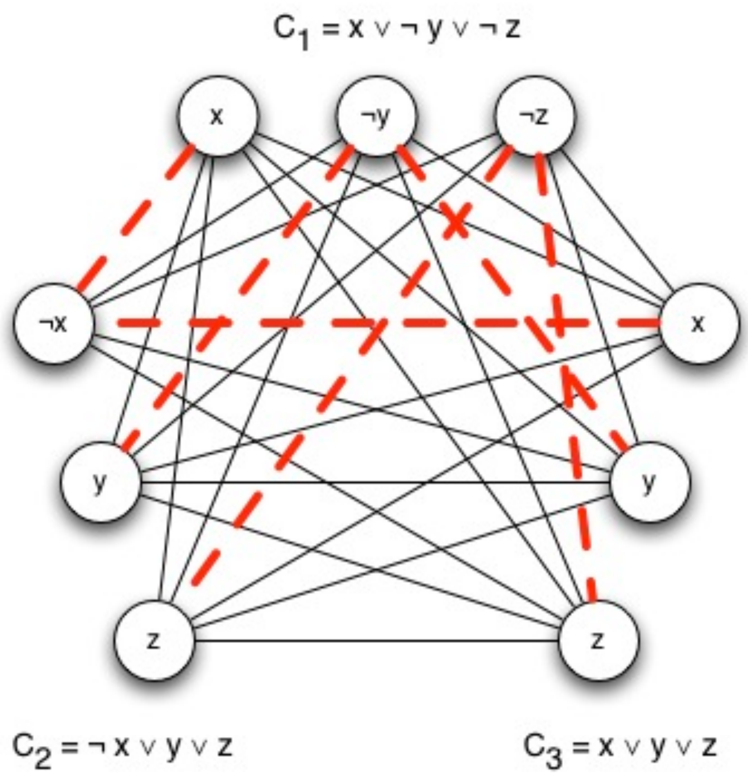
# 3-CNF → k-Clique

- *What should the reduction do?*
- A: Transform a 3-CNF formula to a graph, for which a *k*-clique will exist (for some *k*) iff the 3-CNF formula is satisfiable
- And do this in polynomial time.

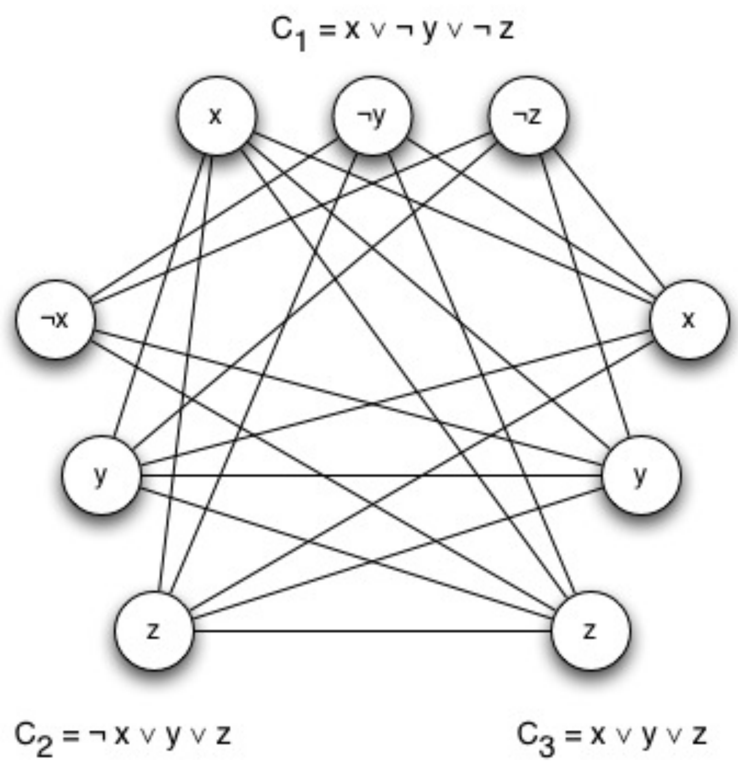
# Reduction: 3-CNF → k-Clique

- Let $B = C_1 \wedge C_2 \wedge \ldots \wedge C_k$ be a 3-CNF formula with *k* clauses, each of which has 3 distinct literals
- For each clause put a triple of vertices in the graph, one for each literal
- Put an edge between two vertices if they are in different triples and their literals are *consistent*, meaning not each other's negation
  - Not consistent: x and ¬x, y and ¬y, etc.
  - Consistent: x and x, x and y, x and ¬y, etc.
- An example:
  $B = (x \vee \neg y \vee \neg z) \wedge (\neg x \vee y \vee z) \wedge (x \vee y \vee z)$
  - See graphs on next pages

# 3-CNF transformed to graph




All edges shown, but those in red connect inconsistent pairs




Just connecting consistent pairs

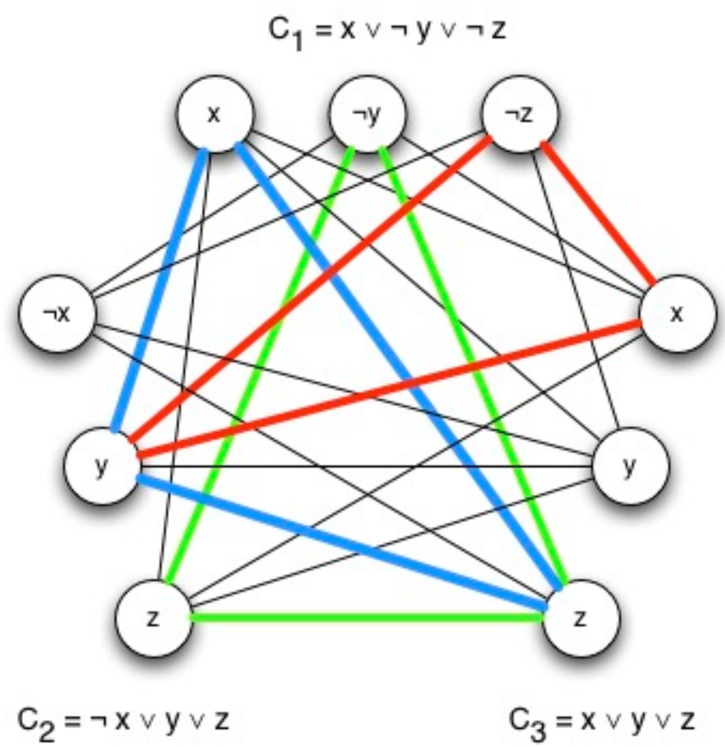
# Graph and Cliques

Each 3-clique is a solution to the 3-CNF instance:

$(x \vee \neg y \vee \neg z) \wedge (\neg x \vee y \vee z) \wedge (x \vee y \vee z)$

- Blue: x true, y true, z true
- Red: z false, y true, x true
- Green: y false, z true, x either
  - Note z=true satisfies both $C_2$ and $C_3$
- Many other 3-cliques

Again, note each 3-clique always has only one node in each clause



# 3-CNF → k-Clique

- Prove the reduction works:
  - If B has a satisfying assignment, then each clause has at least one literal (vertex) that evaluates to 1
  - Picking one such "true" literal from each clause gives a set V′ of *k* vertices. V′ is a clique (*Why?*)
  - If G has a clique V′ of size k, it must contain one vertex in each triple (clause) (*Why?*)
  - We can assign 1 to each literal corresponding with a vertex in V′, without fear of contradiction

# Directed Hamiltonian Cycle ⇒ Undirected Hamiltonian Cycle

- *What was the hamiltonian cycle problem again?*
- For my next trick, I will reduce the *directed hamiltonian cycle* problem to the *undirected hamiltonian cycle* problem before your eyes
  - Why would I want to? To prove something in NP-C
  - *Question: Which variant am I proving NP-Complete?*
- Draw a directed example on the board
  - *Question: What transformation do I need to effect?*

# Transformation: Directed $\Rightarrow$ Undirected Ham. Cycle

- See handout (from page 563 in Baase textbook)
- Transform directed graph G = (V, E) into undirected graph G′ = (V′, E′):
  - Every vertex *v* in V transforms into 3 vertices $v^1$, $v^2$, $v^3$ in V′ with edges $(v^1,v^2)$ and $(v^2,v^3)$ in E′
  - Every directed edge (*v*, *w*) in E transforms into the undirected edge $(v^3, w^1)$ in E′ (draw it)
  - *Can this be implemented in polynomial time?*
  - *Argue that a directed hamiltonian cycle in G implies an undirected hamiltonian cycle in G′*
  - *Argue that an undirected hamiltonian cycle in G′ implies a directed hamiltonian cycle in G*

# Undirected Hamiltonian Cycle

- Thus we can reduce the directed problem to the undirected problem
- *What's left to prove the undirected hamiltonian cycle problem NP-Complete?*
- *Argue that the problem is in* **NP**

# Hamiltonian Cycle ⇒ TSP

- The well-known *traveling salesman problem*:
  - Optimization variant: a salesman must travel to $n$ cities, visiting each city exactly once and finishing where he begins. How to minimize travel time?
  - Model as complete graph with cost c($i,j$) to go from city $i$ to city $j$
- *How would we turn this into a decision problem?*
  - A: ask if $\exists$ a TSP with cost $< k$

# Hamiltonian Cycle $\Rightarrow$ TSP

- The steps to prove TSP is NP-Complete:
  - Prove that TSP $\in$ **NP** (*Argue this*)
  - Reduce the undirected hamiltonian cycle problem to the TSP
    - So if we had a TSP-solver, we could use it to solve the hamilitonian cycle problem in polynomial time
    - *How can we transform an instance of the hamiltonian cycle problem to an instance of the TSP?*
    - *Can we do this in polynomial time?*

# How to show HamCycle $\leq_p$ TSP?

- Transform input for HamCycle into input for TSP
  - HamCycle: Given unweighted graph G1, does it have a ham. cycle?
  - TSP: Given weighted graph G2 and k, is there a ham. cycle with total cost less than k?
- Must convert unweighted graph to weighted
  - Add edges to G1 to make a complete graph G2
  - Add weights as follows:
    - wt(i,j) is 0 if edge i,j is in G1 (original graph for HamCycle)
    - wt(i,j) is 1 if edge i,j is not in G1
- G1 has ham. cycle iff G2 has TSP with k=0
  - Can you see why? Is this transformation polynomial?

# The TSP

- Random asides:
  - TSPs (and variants) have enormous practical importance
    - E.g., for shipping and freighting companies
    - Lots of research into good approximation algorithms
  - Recently made famous as a DNA computing problem
    - “further reading” section of Baase textbook, Ch. 13 (I’ll supply copy if you’re interested)

# General Comments

- Literally hundreds of problems have been shown to be NP-Complete
- Some reductions are profound, some are comparatively easy, many are easy once the key insight is given

# Other NP-Complete Problems

- *Subset-sum*: Given a set of integers, does there exist a subset that adds up to some target $T$?
- *0-1 knapsack*: when weights not just integers
- *Hamiltonian path*
- *Graph coloring*: can a given graph be colored with $k$ colors such that no adjacent vertices are the same color?
- Etc…

# Reminders and Review!

# Important: Input Size and P

- Sometimes a problems seems to be in P but really isn't
- Example: finding if value n is a prime
  - Just loop and do a mod: $\Theta(n)$
- Note that here "n" is not the count or number of data items.
  - There's just one input item.
  - But "n" is a value with a size that affects the execution time.
  - The size of is log(n)
  - T(size) = n but size is log(n).
  - T(log n) = n = $10^{\log n}$ This is really an exponential!
- Be careful if "n" is not a count of data items but a value
  - Dynamic programming problems, e.g. the 0/1 knapsack

# Review (Again)

- A problem B is *NP-complete*
  - if it is in NP **and** it is NP-hard.
- A problem B is *NP-hard*
  - if *every* problem in NP is reducible to **B**.
- A problem A is *reducible* to a problem B if
  - there exists a polynomial reduction function T such that
    - For every string x,
    - if x is a yes input for A, then T(x) is a yes input for B
    - if x is a no input for A, then T(x) is a no input for B.
    - T can be computed in polynomially bounded time.

# NP-Complete Problems

- NP-Complete problems are the "hardest" problems in NP:
  - If any *one* NP-Complete problem can be solved in polynomial time…
  - …then *every* NP-Complete problem can be solved in polynomial time…
  - …and in fact *every* problem in **NP** can be solved in polynomial time (which would show **P = NP**)
  - Thus: solve any NP-Complete problem in $O(n^{100})$ time, you've proved that **P = NP**. Retire rich & famous.

# What We Don't Know: Open Questions

- Is it **impossible** to solve an NP-c problem in polynomial time?
  - No one has proved an exponential lower bound for any problem in NP
  - But, computer scientists believe such a L.B. exists for NP-c problems.
- Are all problems in NP tractable or intractable? I.e., does P=NP or not?
  - If someone found a polynomial solution to any NP-c problem, we'd know P = NP.
  - But, computer scientists believe P≠ NP.

# Unused slides

- Another reduction
  - k-CLIQUE to VertexCover

# Clique → Vertex Cover

- A *vertex cover* for a graph G is a set of vertices incident to every edge in G
- The *vertex cover problem*: what is the minimum size vertex cover in G?
- Restated as a decision problem: does a vertex cover of size *k* exist in G?
- Thm 36.12: vertex cover is NP-Complete

# Clique → Vertex Cover

- First, show vertex cover in **NP** (*How?*)
- Next, reduce *k*-clique to vertex cover
  - The *complement* $G_C$ of a graph G contains exactly those edges not in G
  - Compute $G_C$ in polynomial time
  - G has a clique of size *k* iff $G_C$ has a vertex cover of size |V| - *k*

# Clique $\rightarrow$ Vertex Cover

- Claim: If G has a clique of size *k*, $G_C$ has a vertex cover of size |V| - *k*
  - Let V' be the *k*-clique
  - Then V - V' is a vertex cover in $G_C$
    - Let (*u*,*v*) be any edge in $G_C$
    - Then *u* and *v* cannot both be in V' (*Why?*)
    - Thus at least one of *u* or *v* is in V-V' (*why?*), so edge (*u*, *v*) is covered by V-V'
    - Since true for *any* edge in $G_C$, V-V' is a vertex cover

# Clique $\rightarrow$ Vertex Cover

- Claim: If $G_C$ has a vertex cover $V' \subseteq V$, with $|V'| = |V| - k$, then G has a clique of size $k$
  - For all $u,v \in V$, if $(u,v) \in G_C$ then $u \in V'$ or $v \in V'$ or both (*Why?*)
  - Contrapositive: if $u \notin V'$ and $v \notin V'$, then $(u,v) \in E$
  - In other words, all vertices in V-V' are connected by an edge, thus V-V' is a clique
  - Since $|V| - |V'| = k$, the size of the clique is $k$