CS 4102 — Algorithms
A Polynomial Reduction: 3-COL $\leq_p$ CNF-SAT

Problem: Show that the 3-colorability problem (3-COL) is reducible to the CNF-satisfiability (CNF-SAT) problem. (This, of course, follows from Cook's theorem; give a direct transformation.)

Solution: Find a function $T \in P$ that transforms the input for 3-COL (a graph) into input for CNF-SAT (a logical expression). If $T \in P$ then the number of symbols in the logical expression must be a polynomial function of the number of edges or vertices in the graph.

What sort of CNF logical expression will describe a solution to a graph coloring problem? Each solution to 3-COL must require that:

(1) each vertex has one and only one color;
(2) adjacent vertices do not have the same color.

First we must identify what the logical variables should be. In this case we will need:

x_{ij} vertex i is color j
$\bar{x}_{ij}$ vertex i is **not** color j

How many possible variables will we have? Since there are 3 colors and n vertices, there are a total of $3n$ variables.

Second, we need to specify a number of clauses that will be ANDed together to satisfy the conditions required for a solution for 3-COL. For the moment, consider a complete graph with 4 vertices and $(n^2 - n)/2 = 6$ edges. The required logical expression in this case will need 34 clauses as follows.

The first 16 will be used to satisfy the condition that a vertex can have one and only one color:

$$
\begin{array}{ll}
C(i) := (x_{i1} \vee x_{i2} \vee x_{i3}) & \text{for } i := 1 \text{ to } 4 \\
T(i) := (\bar{x}_{i1} \vee \bar{x}_{i2}) & \text{for } i := 1 \text{ to } 4 \\
U(i) := (\bar{x}_{i1} \vee \bar{x}_{i3}) & \text{for } i := 1 \text{ to } 4 \\
V(i) := (\bar{x}_{i2} \vee \bar{x}_{i3}) & \text{for } i := 1 \text{ to } 4
\end{array}
$$

The four clauses $C(i)$ ensure that each vertex has been assigned at least one color. The four clauses $T(i)$ ensure that both "vertex is color 1" and "vertex is color 2" are **not** true for all vertices. Clauses $U(i)$ and $V(i)$ make sure that a vertex is not assigned two colors for the other two possible pairs of color combinations.

The remaining clauses that are needed ensure that any two vertices connected by an edge do not have the same color. Thus there are 3 for each color for every edge in the graph ($3 \times 6 = 18$ for $n = 4$). Construct a clause $D(e, j)$ for every color j and every edge e connecting vertices u and v as follows:

$$D(e, j) := (\bar{x}_{uj} \vee \bar{x}_{vj})$$

The complete logical expression will be all clauses $C(i), T(i), U(i), V(i)$ and $D(e, j)$ ANDed together.

In general, for n vertices there will be a total of $4n + 3m$ clauses. These will contain $9n + 6m$ literals. Because these are polynomial functions, it is that function T will have a polynomial time complexity based on the size of the input graph. From the above discussion, it is also clear that the graph will be 3-colorable if and only if values can be assigned to the literals so that the CNF logical expression is true. Therefore, we have shown that 3-COL $\leq_p$ CNF-SAT.