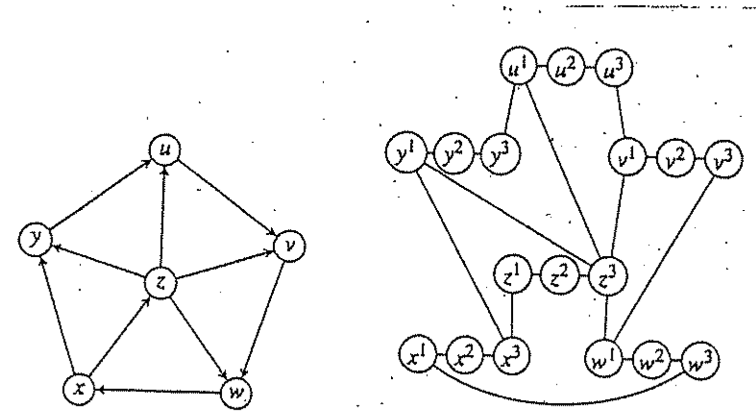
REDUCTION EXAMPLE #2

Theorem 13.7 The directed Hamiltonian cycle problem is reducible to the undirected Hamiltonian cycle problem. (Thus, if we know that the directed Hamiltonian cycle problem is $\mathcal{NP}$-complete, we can conclude that the undirected Hamiltonian cycle problem is also $\mathcal{NP}$-complete.)

Proof Let $G = (V, E)$ be a directed graph with n vertices. G is transformed into the undirected graph $G' = (V', E')$, where, for each vertex $v \in V$, the transformed vertex set V' contains three vertices named v^1, v^2, and v^3. Also, for each $v \in V$, the transformed edge set E' contains *undirected* edges v^1v^2 and v^2v^3. In addition, for each *directed* edge $vw \in E$, E' contains the *undirected* edge v^3w^1. In other words, each vertex of G is expanded to three vertices connected by two edges, and an edge vw in E becomes an edge from the third vertex for v to the first for w. See Figure 13.3 for an illustration. The transformation is straightforward, and G' can certainly be constructed in polynomially bounded time. If $|V| = n$ and $|E| = m$, then G' has $3n$ vertices and $2n + m$ edges.

Now suppose G has a (directed) Hamiltonian cycle $v_1, v_2, \ldots, v_n$. (That is, $v_1, v_2, \ldots, v_n$ are distinct, and there are edges v_iv_{i+1}, for $1 \le i < n$, and v_nv_1.) Then $v_1^1, v_1^2, v_1^3, v_2^1, v_2^2, v_2^3, \ldots, v_n^1, v_n^2, v_n^3$ is an undirected Hamiltonian cycle for G'. On the other hand, if G' has an undirected Hamiltonian cycle, the three vertices, say v^1, v^2, and v^3, that correspond to one vertex from G must be traversed consecutively in the order v^1, v^2, v^3 or v^3, v^2, v^1 since v^2 cannot be reached from any other vertex in G'. Since the other edges in G' connect vertices with superscripts 1 and 3, if for any one triple the order of the superscripts is 1, 2, 3, then the order is 1, 2, 3 for all triples. Otherwise, it is 3, 2, 1 for all triples. Since G' is undirected, we may assume its Hamiltonian cycle is $v_{i_1}^1, v_{i_1}^2, v_{i_1}^3, \ldots, v_{i_n}^1, v_{i_n}^2, v_{i_n}^3$. Then $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$ is a directed Hamiltonian cycle for G. Thus G has a directed Hamiltonian cycle if and only if G' has an undirected Hamiltonian cycle. □

It is of course much easier to see that the G' defined in the proof is the proper transformation to use than it is to think up the correct G' in the first place, so we make a few observations to indicate how G' was chosen. To ensure that a cycle in G' corresponds to a cycle in G, we need to simulate the direction of the edges of G. This aim suggests giving G' two vertices, say v^1 and v^3, for each v in G with the interpretation that v^1 is used for edges in G whose head is v and v^3 is used for edges whose tail is v. Then wherever v^1 and v^3 appear consecutively in a cycle for G' they can be replaced by v to get a cycle for G, and vice versa. Unfortunately, there is nothing about G' that forces v^1 and v^3 to appear consecutively in all of its cycles; thus G' could have a Hamiltonian cycle that does not correspond to one in G (see Exercise 13.13). The third vertex, v^2, which can only be reached from v^1 and v^3, is introduced to force the vertices that correspond to v to appear together in any cycle in G'.



$G = (V, E)$ $G' = (V', E')$

Figure 13.3 Reduction of the directed Hamiltonian cycle problem to the undirected Hamiltonian cycle problem.