Homework #1: Set Theory, Proofs, Finite Automata
Due Date: Thursday, 7 September 2006

Word to the wise. Do *not* start these problem sets at the last minute. Do *not* wait for the lecture about DFA's to start thinking about the DFA problems. There is a reason you have a textbook.

Problem 1. [6 points]

True or false:

(a) $\emptyset \subseteq \emptyset$

(b) $\emptyset \subset \emptyset$

(c) $\emptyset \in \emptyset$

(d) $\{1,2\} \in 2^{\{1,2\}}$

(e) $\{x,y\} \in \{\{x,y\}\}$

(f) $\{x,y\} \subseteq \{\{x,y\}\}$

Problem 2. [8 points]

Which of the following sets is closed under the given operation? Prove your answer.

(a) $\{x|x \text{ is an odd integer}\}$, multiplication

(b) $\{x|x = 2n, n \in \mathbb{Z}\}$, subtraction

(c) $\{2m+1|m \in \mathbb{Z}\}$, division

(d) $\{z|z = a+bi, a,b \in \mathbb{R}, i = \sqrt{-1}\}$, exponentiation

Problem 3. [10 points]

Is there a bijection between $\{x|x \in \mathbb{R}, 0 < x < 1\}$ and $\mathbb{R}$? If not, prove it. If so, give one.

Problem 4. [10 points]

What is the cardinality of each of these sets?

(a) The set of all polynomials with rational coefficients

(b) The set of all functions $f : \mathbb{R} \to \mathbb{R}$

(c) The set of all possible Python programs

(d) The set of all finite strings over the alphabet $\{0, 1, 2\}$

(e) The set of all 5×5 matrices over the rationals

(f) The set of all points in 3-dimensional Euclidean space.

(g) The set of all English words

(h) $\{\emptyset, \mathbb{N}, \mathbb{Q}, \mathbb{R}\}$

(i) $\mathbb{N} \times \mathbb{Z} \times \mathbb{Q}$

(j) $\mathbb{R} - \mathbb{Q}$

Problem 5. [10 points]

Show that $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z} | n^4 - 4n^2 = 3k$.

Problem 6. [10 points]

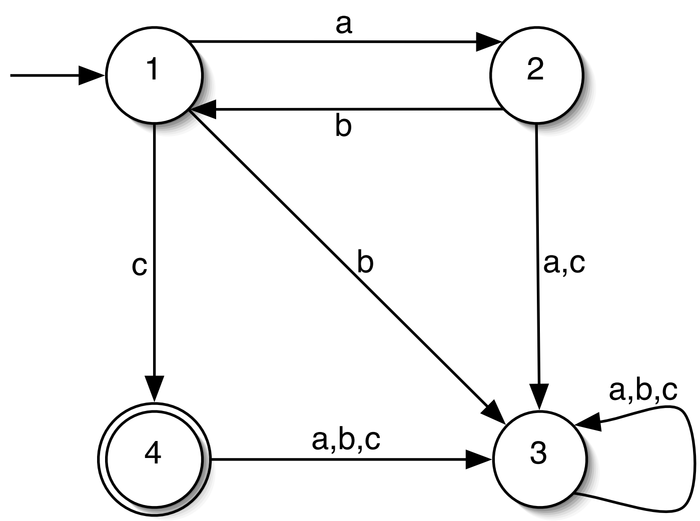
Show that in any group of people, there are at least two people with the same number of acquiantances within the group. Assume that the "acquiantance" relation is symmetric but not reflexive.

Problem 7. [15 points]

Prove that $|\{p | p \text{ prime}\}| = \aleph_0$.

Problem 8. [10 points]

Consider the DFA $M = (Q, \Sigma, \delta, q_0, F)$ with $Q = \{1, 2, 3, 4\}$, $\Sigma = \{a, b, c\}$, $q_0 = 1$, $F = \{4\}$, and δ as defined in the following transition diagram:



Give a concise, informal description of the language of this DFA.

Problem 9. **[20 points]**

Construct DFAs for the following languages over the alphabet $\Sigma = \{0, 1\}$:

(a) The set of strings which do not contain the substring 101.

(b) The set of strings in which the number of 0's is divisible by 2 and the number of 1's is divisible by 3.

Problem 10. **[25 points]**

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA, and $a \in \Sigma$ be a particular alphabet symbol such that:

$$\forall q \in Q, \delta(q, a) = q.$$

(a) [20 points] Prove that for all $n \geq 0$ and for all $q \in Q$,

$$\hat{\delta}(q, a^n) = q.$$

(b) [5 points] Show that either $\{a\}^* \subseteq L(M)$ or $\{a\}^* \cap L(M) = \emptyset$.

Problem 11. **[30 points]**

(a) [5 points] Let $L \subset \{0, 1\}^*$ be the language of all strings such that there are two 0's separated by a number of positions that is a multiple of 5 (including zero). For example, 1010110 is not in L, but 10110110 is in L. Construct an NFA for this language.

(b) [25 points] You should be glad that I did not ask you to construct a DFA for this language. To truly appreciate this, prove that any DFA for this problem must have at least 2^5 states. (*Hint*: Why does your argument fail for NFAs? How does the lower bound on the DFA size compare with your NFA's size? What are the implications for the relative power of NFAs and DFAs?)

Problem 12. [20 points]

Consider the NFA $N = (Q, \Sigma, \delta, q_0, F)$ which has $Q = \{q_0, q_1, q_2, q_3, q_4\}$, $\Sigma = \{0, 1\}$, $F = \{q_3, q_4\}$, and the transition function as defined in the following table:

δ	0	1
q_0	$\{q_0, q_1\}$	$\{q_0\}$
q_1	$\{q_2, q_3\}$	$\{q_4\}$
q_2	$\{q_0, q_2\}$	$\{q_4\}$
q_3	$\emptyset$	$\emptyset$
q_4	$\emptyset$	$\emptyset$

(a) Convert this NFA into a DFA using the subset construction described in class. Your solution should consist of a transition diagram with *only* the essential states.

(b) Give a concise, informal description of the language defined by this DFA.