Homework #3: Context Free Languages and Pushdown Automata
Due Date: Tuesday, October 10, 2006

Problem 1. [15 points]

Consider the context-free grammar $G = (V, T, P, S)$, where $V = \{A, B, S\}$, $T = \{0, 1\}$, and the productions are:

$$\begin{aligned} S &\rightarrow 0A \mid 0S0 \mid 0B0 \\ A &\rightarrow S0 \\ B &\rightarrow 1B \mid 1 \end{aligned}$$

(a) Give a brief description of the language of this grammar.

(b) Show that this grammar is ambiguous by demonstrating that a string in $L(G)$ has two distinct leftmost derivations

(c) Specify which productions you will drop from this grammar to make it unambiguous *without changing its language*. Briefly justify your answer.

Problem 2. [25 points]

Give a context free grammar for the language L consisting of all strings from $\{0, 1\}^*$ which have exactly twice as many 0's as 1's. For example, $011000 \in L$, but $1011000 \notin L$.

Problem 3. [20 points]

We have discussed the issue of ambiguity of CFGs for arithmetic expressions written in *infix* notation (where binary operators are written between the two operands—e.g., $x + y$ or $x \times y$). There are other ways of writing operators, such as the *prefix* notation where we would write $+xy$ or $\times xy$, and the *postfix* notation where we would write $xy+$ or $xy\times$, to denote the same arithmetic expressions.

The following context-free grammar generates arithmetic expressions in the postfix notation (which is used, for example, in the programming language APL). Notice the lack of parentheses in this grammar.

$$S \rightarrow SS+ \mid SS- \mid SS\times \mid x \mid y$$

(a) [5 points] Find a derivation tree for the string $xxy \times x - +$.

(b) [15 points] Is this grammar ambiguous? Give a brief justification. While a formal proof is not required, you should think about how you would prove this formally.

Problem 4. [15 points]

Consider the PDA M with the following transitions (assume that q_0 is a final state):

$$\begin{aligned}\delta\,(q_0, 0, Z_0) &= \{(q_0, AZ_0)\}\\ \delta\,(q_0, 0, A) &= \{(q_0, AA)\}\\ \delta\,(q_0, 1, A) &= \{(q_0, \epsilon)\}\end{aligned}$$

(a) Give an execution trace (using instantaneous descriptions) of the PDA M showing that input string 001011 is in $L(M)$.

(b) Give a succinct description of the language $L(M)$.

(c) Suppose we add the transition

$$\delta\,(q_0, \epsilon, Z_0) = \{(q_0, \epsilon)\}$$

to this PDA. Give a succinct description of the *empty stack language* $N(M)$.

Problem 5. [15 points]

Recall that a normal PDA removes the top symbol from the stack just before each transition, although of course it can push this symbol back onto the stack if desired. Define a "push back" PDA (PB-PDA) to be a PDA which, during each transition, is *required* to push back the current top-of-stack symbol before making the transition. Another way to think about this kind of machine is that it never pops the top-of-stack symbol in the first place.

What is the class of languages that can be accepted as a *final state language* by PB-PDAs? Justify your answer.

Problem 6. [10 points]

Convert the grammar

$$\begin{aligned}S &\rightarrow aAA\\ A &\rightarrow aS \mid bS \mid a\end{aligned}$$

to a PDA that accepts the same language by empty stack.

Problem 7. [20 points]

Consider the context free grammar

$$
\begin{aligned}
S &\rightarrow AAA \mid B \\
A &\rightarrow aA \mid B \\
B &\rightarrow \epsilon
\end{aligned}
$$

Perform the following operations (in the given order) on this grammar, showing the new grammar after each step:

(a) Remove ϵ-productions.

(b) Remove unit productions.

(c) Remove useless symbols.

(d) Convert to Chomsky Normal Form.

Problem 8. [20 points]

Show that the following language is not context free.

$$L = \{a^n b^m a^n b^m \mid n, m \geq 0\}$$

Problem 9. [20 points]

Show that the following language is not context free.

$$L = \{0^i 1^j \mid j = i^2\}$$