Homework #5: Decidability and Turing Machines
Due Date: Tuesday, November 16, 2006

Problem 1. [10 points]

In this problem you will establish some closure properties for languages of Turing machines. The proof or justification should be at the same high level as my proofs for closure under union in class.

(a) Are recursive languages closed under intersection?

(b) Are recursively enumerable languages closed under intersection?

Problem 2. [15 points]

Show that the set

$$\{\langle M \rangle | M \text{ writes a 1 somewhere on the tape}\}$$

is undecidable.

Problem 3. [15 points]

Show that the set

$$\{\langle M \rangle | \text{ when started with a blank tape, } M \text{ will write a nonblank somewhere}\}$$

is decidable.

Problem 4. [20 points]

Consider the following language:

$$L_k = \{\langle M_1, M_2, k \rangle | \text{ for Turing Machines } M_1 \text{ and } M_2, |L(M_1) \cap L(M_2)| \geq k\}.$$

That is, all encodings of Turing machines M_1, M_2, and a positive integer k, such that there are at least k strings that are accepted by both machines.

(a) [5 points] Show that L_k is recursively enumerable.

(b) [15 points] Show that L_k is undecidable. You may use the fact that the language

$$L_e = \{\langle M \rangle | \text{ for Turing Machine } M, L(M) = \emptyset\}$$

is *not* recursively enumerable.

Problem 5. [20 points]

Consider the following two languages over the alphabet $\Sigma = \{0, 1\}$.

$$\begin{aligned} L_U &= \{\langle M, w\rangle | \text{ Turing machine } M \text{ accepts input } w\} \\ L_{15} &= \{\langle \hat{M}\rangle | \text{ for Turing machine } \hat{M}, |L(\hat{M})| = 15\} \end{aligned}$$

The language L_U is the universal language, while the language L_{15} corresponds to the following decision problem: given a Turing Machine $\hat{M}$, does its language have size exactly 15?

Your goal is to show that L_{15} is undecidable, assuming that L_U is undecidable. We propose the following reduction from L_U to L_{15}:

Given a Turing Machine M and input w, construct a Turing Machine $\hat{M}$ which behaves as follows on being given input $\hat{w}$:

1. $\hat{M}$ simulates the behavior of M on input w
2. if M halts on w and accepts, then $\hat{M}$ examines its own input $\hat{w}$, halting in a final state if $|\hat{w}| \leq 3$ and halting in a non-final state otherwise
3. if M halts on w and rejects, then $\hat{M}$ rejects its own input $\hat{w}$.

(a) [5 points] In the case where $w \in L(M)$, what is $L(\hat{M})$?

(a) [5 points] In the case where $w \notin L(M)$, what is $L(\hat{M})$?

(a) [10 points] Using the reduction described above, show that L_{15} is undecidable.

Problem 6. [20 points]

Recall that Rice's Theorem states that every nontrivial property of the recursively enumerable languages is undecidable. For each of the languages given below, choose one of the following three statements which best describes the situation:

- The language is decidable
- The langauge is undecidable and this follows from a *direct* application of Rice's Theorem.
- The language is undecidable, but we cannot directly apply Rice's Theorem to show this.

Briefly justify your answer. Indicate your choice by writing A, B, or C for each language description. Note that in all cases M refers to a Turing Machine.

1. $L = \{\langle M\rangle | L(M) \text{ is context free}\}$.

2. $L = \{\langle M \rangle | L(M) \text{ is recursively enumerable}\}$.
3. $L = \{\langle M \rangle |$ when started on a blank tape, Turing machine M eventually halts in a final state$\}$
4. $L = \{\langle M_1, M_2 \rangle | L(M_1) = L(M_2)\}$
5. $L = \{\langle M, w \rangle | M$ accepts w in less than 100 steps$\}$