Closure Systems and their Structure

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Closure is a fundamental property of many discrete systems. Transitive closure in relations has been well studied, *e.g.* [1,14,6,5], as has geometric closure [8,9] and closure in various kinds of graphs [17,10]. The closed sets of a closure operator illustrate a kind of well-behaved internal structure that is the main theme of this paper.

In Section 1, we examine antimatroid closure spaces. In Section 2, we consider a closure operator that has been widely used in digital image processing [25]. This operator, which can be equally well defined on graphs, is not antimatroid; but it is shown in Section 3 that it retains many of the same structural properties, and is closely related to the classic graph-theoretic theme of domination. Finally, in Section 4, we relate these concepts to premise system[30].

1 Antimatroid Closure Spaces

Assume we have a closure operator φ satisfying the usual axioms: $\forall Y \subseteq \mathbf{U}$ (we let $\mathbf{U}$ denote the universe, or entire space)

$Y \subseteq Y.\varphi$,

$X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$, and

$Y.\varphi.\varphi = Y.\varphi$.

(Notice that we denote all set valued operators with Greek characters, using a postfix dot notation.) A set Y is closed if $Y = Y.\varphi$. Closure is preserved under intersection. A minimal set $Y.\kappa$ of elements that will generate $Y.\varphi$, that is $Y.\kappa.\varphi = Y.\varphi$, is called, depending on one's discipline, the **generators**, kernel, basis, or extreme points of the set.

Closure operators are fundamental in much of mathematics. It is common to assume the Steinitz-MacLane exchange axiom

[1] Research supported in part by DOE grant DE-FG05-95ER25254.

$$\text{if } p, q \notin Y.\varphi \text{ then } q \in \{Y \cup p\}.\varphi \text{ implies } p \in \{Y \cup q\}.\varphi$$

in which case the resulting system is called a **matroid** [29]. If **U** is a set of n-dimensional vectors and φ is taken to be the spanning operator, then each closed set is a vector space. A minimal set of generating vectors is a basis for the vector space. The term matroid is derived from the traditional representation of these linear spaces as matrices.

Although matroids are quite general, for example the circuits of an undirected graph constitute a matroid, the authors have been particularly interested in closure systems which satisfy an *anti*-exchange axiom. A closure system satisfying

$$\text{if } p, q \notin Y.\varphi \text{ then } q \in \{Y \cup p\}.\varphi \text{ implies } p \notin \{Y \cup q\}.\varphi$$

is called an **antimatroid** or equivalently, because it is easy to show that $Y.\kappa$ must be unique, a **uniquely generated closure space**. Such antimatroid closure spaces are most often encountered as convex geometries [18,8,9] in the literature. But, antimatroid closure operators abound. In [21], it was shown that n^n unique closure operators can be defined on any n element set, $n \geq 10$. They have been studied in chordal and block graphs [17,10] and in partial orders [22]. Antimatroid closure spaces are important because they support greedy algorithms, many of which are examined as shelling operators in [19].

In [23], it was shown that the subsets of any closure space $(\mathbf{U}, \varphi)$ could be ordered by

$$X \leq_\varphi Y \quad \text{iff} \quad Y \cap X.\varphi \subseteq X \subseteq Y.\varphi. \tag{1}$$

The resulting partial order on all the subsets of **U** is a lattice $\mathcal{L}$ on the $2^{|U|}$ elements of the power set, as shown in Figure 1(b) provided φ is antimatroid. Here the sublattice of closed subsets, $\mathcal{C} = [\emptyset, bcdef]$, is connected with solid lines, generally oriented from lower left to upper right. The set $[Y.\varphi, Y.\kappa]$ of all sets Y with the same closure $Y.\varphi$ is easily shown to be a distributive boolean algebra with $Y.\kappa$ as its supremum. These are illustrated using bolder dashed lines oriented from lower right to upper left.

Many individuals have independently discovered that the sublattice $\mathcal{C}$ of closed sets is semi-modular [20], and if every singleton is closed it is atomic as well. The entire lattice $\mathcal{L}$ is neither semi-modular nor complemented nor atomic; but it is very "regular". Two properties we will use in this paper were demonstrated in [23].

Theorem 1 (Fundamental Covering Theorem) *If $p \notin X$ then*

(a) $X \leq_\varphi X \cup \{p\}$ if and only if $p \notin X.\varphi$

(b) $X \cup \{p\} \leq_\varphi X$ if and only if $p \in X.\varphi$

where

(a) is a cover if and only if $(X \cup \{p\}).\varphi \subseteq X.\varphi \cup \{p\}$, and

(b) is always a covering relationship.

Moreover, if φ is uniquely generated then (a) and (b) characterize all covering relations in $(2^{\mathbf{U}}, \leq_\varphi)$.

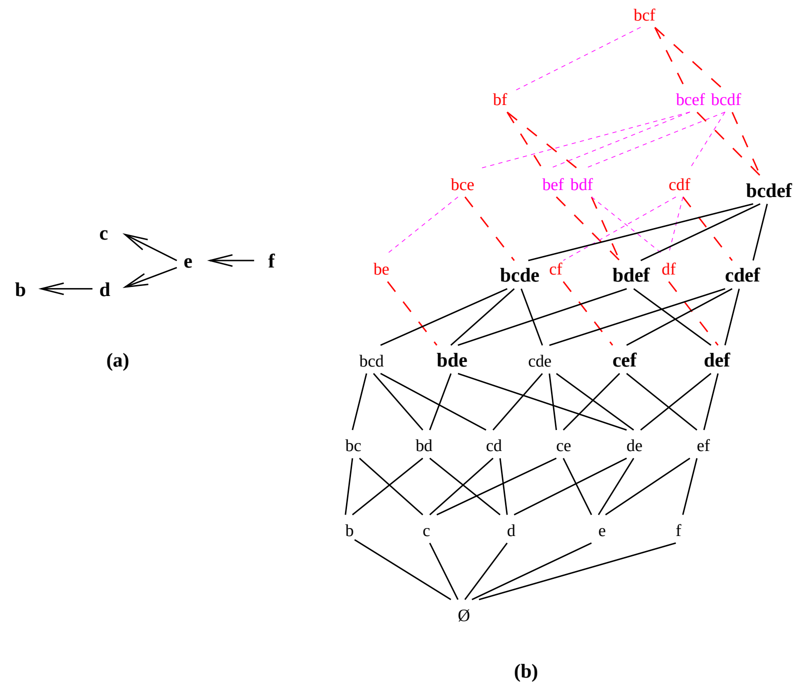

Fig. 1. An acyclic graph (a), with convex closure lattice $\mathcal{L}$ (b)

Theorem 2 (Fundamental Structure Theorem) *Let* $X.\varphi \leq_\varphi Y.\varphi$ *and let* $X \in [X.\varphi, X.\kappa]$. *There exists a unique* $Y \in [Y.\varphi, Y.\kappa]$ *such that* $X \leq_\varphi Y$, *where* Y *is minimal wrt.* $\leq_\varphi$ *(maximal wrt.* $\subseteq$*). Moreover* $Y = X \cup \Delta$ *where* $\Delta = Y.\varphi - X.\varphi$ *and* $Y = Y.\varphi - \delta$ *where* $\delta = X.\varphi - X$.

These two properties are seen in Figure 1(b). Consider $Y.\varphi = bdef$ which is generated by bf. Both $Y.\varphi - b = def$ and $Y.\varphi - f = bde$ are closed; and $[bdef, bf]$ is replicated in (actually isomorphic to) $[bcdef, bcf]$.

Convex operators such as a convex hull operator are usually uniquely generated. Figure 1(b) arose from applying the convex closure operator $Y.\varphi = \{y | y_1 \leq y \leq y_2, y_i \in Y\}$ to the acyclic graph of Figure 1(a). A five point discrete planar geometry in which convexity is determined by Euclidean measure is shown in Figure 2. Both of these are exceedingly small five point systems to facilitate illustration; but the basic properties are apparent.

2 Pseudo Convexity in Images

A basic closure operator in any n-dimensional space is the convex hull operator. The discrete geometry of Figure 2 is representative. However in a pixel image, defining a notion of convexity that is completely conformal to Euclidean convexity is difficult [11,4]. But a pseudo convex hull operator of a set of pixels based on the alternate expansion to adjacent pixels, followed by a similar contraction is both easy to define

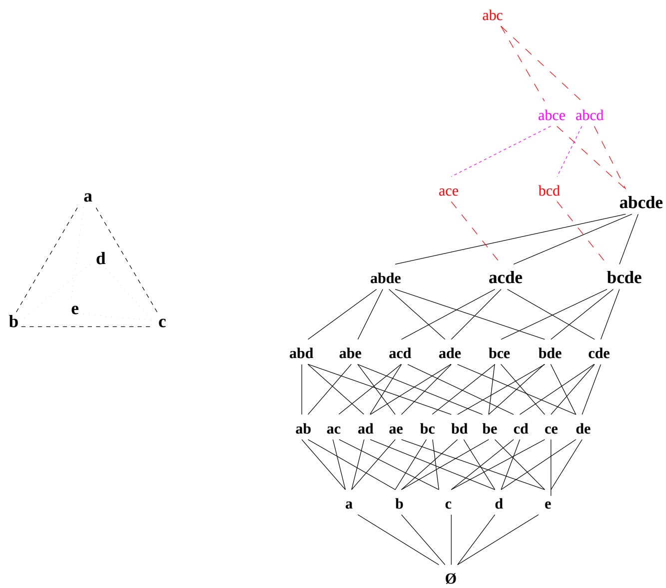

Fig. 2. An discrete geometry (a), with convex closure lattice $\mathcal{L}$ (b)

and to implement [25]. [2] We illustrate the single step version in Figure 3 assuming a 4 neighbor adjacency has been around for many years and has been widely used for local smoothing. Readily, Figure 3(c) is not a convex hull of Figure 3(a), regardless

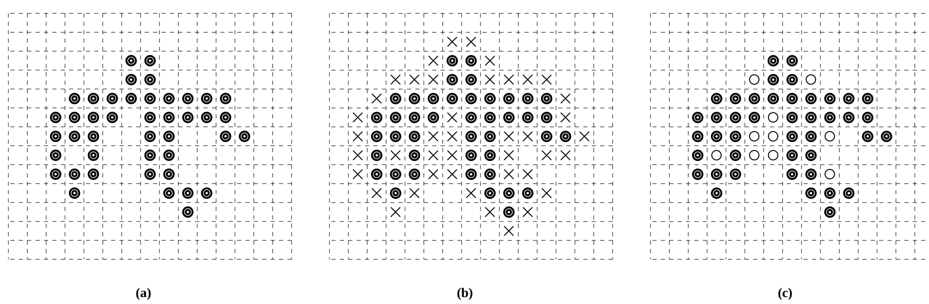

Fig. 3. A discrete pseudo convex hull operator,

of how one defines "convexity". But, Figure 3(c) is closed in the sense that it satisfies the three closure axioms. In particular, once again expanding and contracting by 1 step will add nothing to the figure. We prove this idempotency in the next section. We call this coarse approximation **pseudo convexity**. One can use more than a single step; and one can alternate 4 and 8 neighbor cycles to get increasingly better approximations of a Euclidean convex hull [26].

That such a process generates, in the limit, a convex hull can be most easily shown by analogy in R^2 as in Figure 4. It is readily apparent that expanding a region by ϵ then shrinking it by ϵ will generate a true convex hull as $\epsilon \to \infty$. A more rigorous argument can be made by observing that the area A is non-decreasing while the perimeter is non-increasing as $\epsilon \to \infty$. In the messier discrete world, one must show

[2] One can also expand *inwards* from the boundary of a pixel set. This has been used to capture the anti-convex structure in the form of a "Blum" skeleton [24,27].

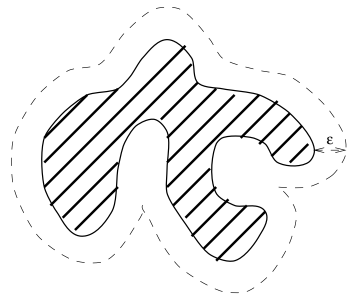

Fig. 4. Expansion of a planar region by ϵ,

that expansion, followed by contraction of k steps, where $k \geq \lfloor d/2 \rfloor$, $d =$ the diameter of the largest subset, will result in hulls H_i having the property that $H_i \cap H_j$ is still convex and that each H_i contains all closed sets generated by any two points of the set. The H_i need not visually appear to be "convex"; but they will have the necessary properties.

3 Pseudo Convexity in Graphs

In [12], the authors emphasize the role of the "local subset problem". This approach is as unusual in graph theoretic studies as it is common in image analysis. We regard it as a fundamental point of departure. Let $G = (N, E)$ be any undirected graph on the set N of nodes and E of edges. For any $Y \subseteq N$ we let $Y.\eta$ denote the open neighborhood of Y, that is $Y.\eta = \{z \notin Y | \exists y \in Y \wedge (y, z) \in E\}$, and let $Y.\bar{\eta}$ denote the **closed neighborhood**, or $Y.\eta \cup Y$. [3] By the **neighborhood closure**, φ_η we mean the set $Y.\varphi_\eta = \{z | z.\bar{\eta} \subseteq Y.\bar{\eta}\}$. Notice, that this closure concept precisely captures the process of expansion and contraction of the preceding section because $z \in Y.\varphi$ if and only if each neighbor of z is a neighbor of Y, and hence "filled" when Y is expanded.

Lemma 3 *φ_η is a closure operator.*

PROOF. It is apparent that $Y \subseteq Y.\varphi_\eta$ and $X \subset Y$ implies $X.\varphi_\eta \subseteq Y.\varphi_\eta$. Only idempotency is questionable because, in general, $Y.\bar{\eta} \subset Y.\bar{\eta}.\bar{\eta}$.
Let $y \in Y.\varphi_\eta.\varphi_\eta$ and suppose $y \notin Y.\varphi_\eta$. The latter implies $\exists z \in y.\bar{\eta}$ such that $z \notin Y.\bar{\eta}$. But, $y \in Y.\varphi_\eta.\bar{\eta}$ requires that $\exists y' \in Y.\varphi_\eta$ such that $z \in y'.\bar{\eta}$. However, this implies $y' \notin Y.\varphi_\eta$ (for the same reason that $y \notin Y.\varphi_\eta$) and contradiction. So $y \in Y.\varphi_\eta$ and $Y.\varphi_\eta.\varphi_\eta \subseteq Y.\varphi_\eta$. $\square$

In Figure 5, we have a small 8 element graph with 26 subsets closed under φ_η as shown in Figure 5(b). φ_η is clearly not a uniquely generated closure. We see that $\{egh\}.\kappa =$

[3] In the graph theory literature, open neighborhoods are often denoted by $N(Y)$ and closed neighborhoods by $N[Y]$ [2,12].

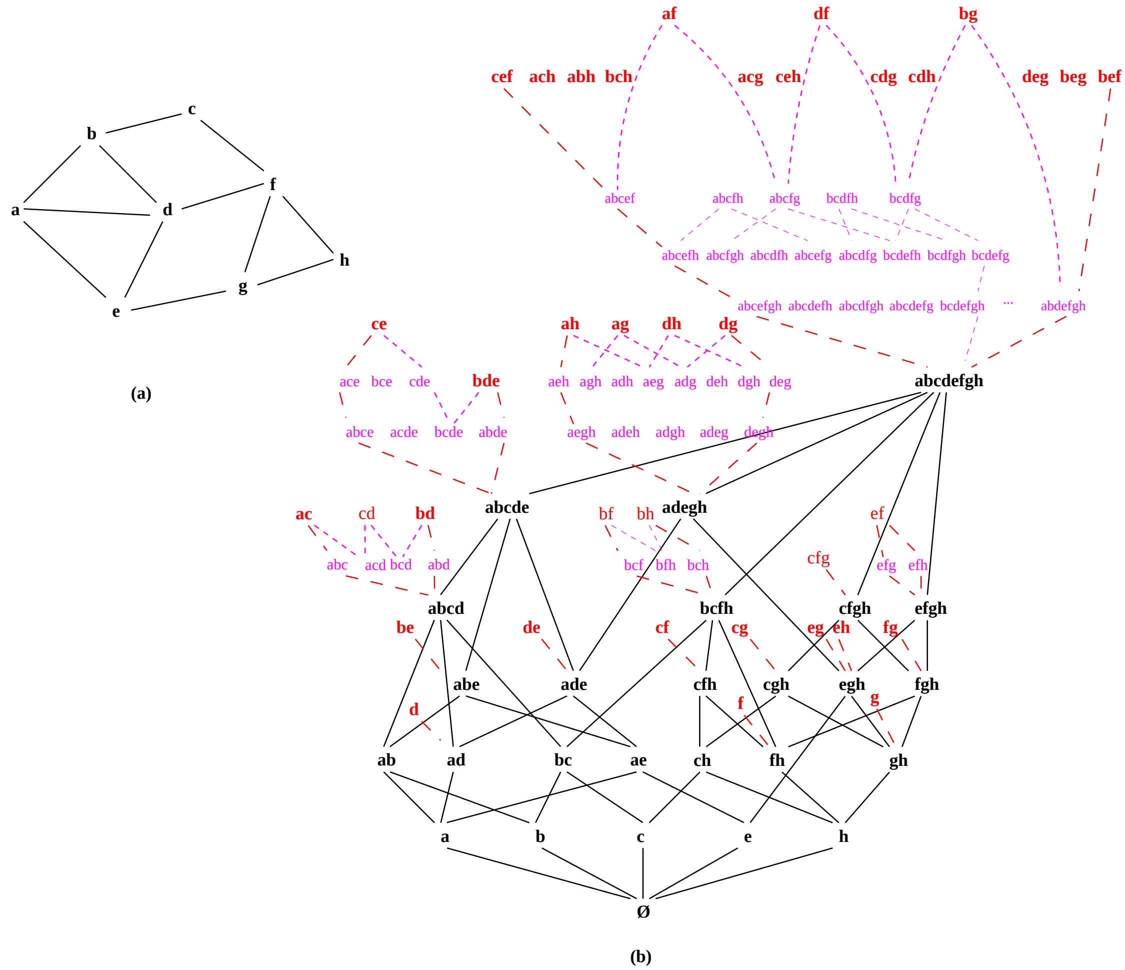

Fig. 5. Pseudo Convexity Operator

$\{\{eg\},\{eh\}\}$, $\{abcde\}.\kappa = \{\{ce\},\{bde\}\}$, and $\{abcdefgh\}.\kappa = N.\kappa$ consists of 14 minimal generators. We have only sketched in a few of the 171 subsets in the interval $[N, N.\kappa]$ to suggest its structure. The subsets of N, partially ordered by (1), are *not* a lattice.

But, many of the important properties of closure lattices still hold. We observe that each of the structures $[Y.\varphi_\eta, Y.\kappa]$[4] is replicated in $[Z.\varphi_\eta, Z.\kappa]$ where $Y.\varphi_\eta \leq_\varphi Z.\varphi_\eta$. Theorem 2 remains true, even if the closure space is not uniquely generated The reader should verify that $[\{abcde\}, \{abcde\}.\kappa]$ is similarly replicated with $\Delta = \{fgh\}$.

But, for non-uniquely generated closure spaces, Theorem 1 is no longer valid; it must be generalized. To do this we use the concept of a blockers in a graph. Let $\mathcal{F}$ be any family of sets. A set B is said to be a **blocker** for $\mathcal{F}$ if $\forall X \in \mathcal{F}, B \cap X \neq \emptyset$.

[4] Because φ_η is not uniquely generated, these intervals are not Boolean algebras.

Theorem 4 (Generalized Fundamental Covering Theorem) *Let Y be closed and let $Y.K = \{Y.\kappa\}$ be its family of minimal generators.*

(a) If $X \subset Y$ and X is closed, then $Y-X$ is a blocker of $Y.K$.

(b) If B is a minimal blocker of $Y.K$, then $Y-B$ is closed.

(c) Y covers X in $\mathcal{L}_\varphi$ iff $Y-X$ is a minimal blocker of $Y.K$.

PROOF.

(a) Let $Y.\kappa \in Y.K$ and suppose $Y.\kappa \cap (Y-X) = \emptyset$. Then, since $Y.\kappa \subseteq Y, Y.\kappa \subseteq X$. But, $Y.\kappa.\varphi = Y$ and thus $Y \subseteq Y.\kappa.\varphi \subseteq X.\varphi = X$, a contradiction.

(b) Let $Z = (Y-B).\varphi$. Then $Z \subset Y.\varphi = Y$. If $Z = Y$, then $Y-B$ is a generating set for Y, so it contains some minimal generating set $Y.\kappa$. Now, $Y.\kappa \subseteq Y-B$ implying $Y.\kappa \cap B = \emptyset$, contradicting assumption that B is a blocker. So $Z \neq Y$.
Since Z is closed and $Z \subset Y$, by (a) $Y-Z$ is a blocker of Y. Because $Y-Z$ is a blocker, and because $Y-Z = Y-(Y-B).\varphi \subseteq Y-(Y-B) = B$, and because B is a minimal blocker, we have $B = Y-Z$. Thus $Z = Y-B$, and because Z is closed, $Y-B$ must be as well.

(c) readily follows from (a) and (b). If Y covers X in $\mathcal{L}_\varphi$, then $Y-X$ is a minimal blocker of $Y.K = \{Y.\kappa\}$; and if B is a minimal blocker of $Y.K$, then $X = Y-B$ is closed and Y covers X. □

That is, provided Z is not the whole space, we may pick an element from each of its generating sets (subject to the constraint that the elements are distinct and do not themselves constitute a generating set). Deletion of such a set $\Delta = \cup X_i$ from Z will yield another closed set that will be covered by Z with respect to φ. Observe, that from the 4 generating sets of $\{adegh\}.\kappa = \{\{ah\},\{ag\},\{dh\},\{dg\}\}$ one may choose $\Delta = \{ad\}$ or $\Delta = \{gh\}$; but no others.

Figure 5 is only a small system. It is worthwhile to look at two closed subsets of a 26 element graph, as in Figure 6.[5] Observe that $\{egp\}.\varphi_\eta = \{befgjkp\} = Z_1$ and that $\{jotv\}.\varphi_\eta = \{jkopstuvwyz\} = Z_2 = \{joxv\}.\varphi_\eta = \{joyv\}.\varphi_\eta$. Their intersection $\{jkp\}$ is closed, with a unique generator {jp}. Smaller closed sets can be identified using the fundamental covering theorem. For instance, $Z_1-\{g\}$ is closed. So is $Z_2-\{j\}$, because j is an element of every generator, and once chosen as X_1, we have the unique element from all other κ_i. $Z_2-\{x\}$ is not closed, however $Z_2-\{txy\}$ is. The important observation is that any result regarding closure on an arbitrary graph must be true on the regular rectlinear or hexagonal mesh. Important components of any computer scene are likely to be "closed", if not convex [13]. These results indicate how these closed subsets can be structured and later traversed.

[5] Both graphs are planar to simplify their drawing. But, there is no reason to exclude non-planar graphs or tesselations of n-space.

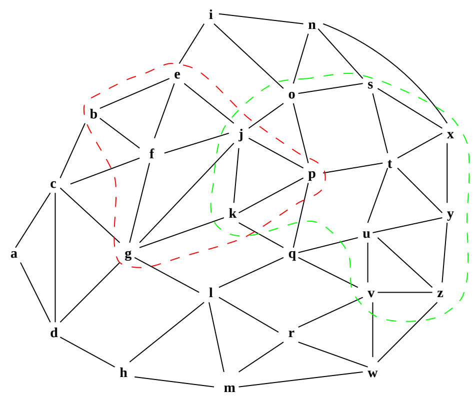

Fig. 6. Two Pseudo Convex Subsets

A subset Y is said to dominate $G = (N, E)$ if $Y.\bar{\eta} = N$. If Y is a minimal set with this property, we say Y is a **domination set**. The concept of dominance is one of the oldest, and still one of the most vital, themes in abstract graph theory. Its origins come from the issues of chessboard domination by various pieces, *e.g.* the 8-queens problem; today, characteristic graph parameters and algorithmic complexity are centers of attention [28,15,12]. For example, $\gamma(G) \leq \Gamma(G)$ denote the size of the smallest and largest minimal (irredundant) dominating set of G.

In [3], Berge and Duchet conduct an investigation of a variety of generating concepts, which they call *kernels*. These dominate the graph. Observe that each of the generating sets of $N.\kappa$ of Figure 5 is a dominating set. We observe that $\gamma(G) = 2 < 3 = \Gamma(G)$ in this graph. Theorem 2 provides a kind of upward mobility through any closure system, by which identification of smaller closed subsets and their generating subsets can be used to find larger closed sub-structures. Just as we are investigating this property in the case of closed, pseudo convex subsets of an image, we are also investigating this as a way of finding dominating sets in arbitrary graphs.

4 Premise Systems and Bases

Let φ be a closure operator on a finite set $\mathbf{U}$. A family $\mathcal{PS}$ of subsets of $\mathbf{U}$ is a **premise system** for φ [16] if

$$Y \subseteq \mathbf{U} \quad \text{is closed, if and only if} \quad \forall P \in \mathcal{PS}, P \subseteq Y \quad \text{implies} \quad P.\varphi \subseteq Y. \quad (2)$$

We can make a few immediate observations. First, lets assume that $\emptyset$ is closed. Inclusion of any closed set X provides no discrimination since one only has $X \subseteq Y$ iff $X.\varphi = X \subseteq Y$. So only non-closed sets are interesting candidates for $\mathcal{PS}$. Let X be a minimal (wrt. set inclusion) non-closed set, $X \in \mathcal{PS}$ because otherwise (2) is satisfied vacuously, implying X is closed. In Figure 1(b), these are the sets $\{be\}, \{cf\}$, and $\{df\}$, which also happen to be generating sets.

It is worthwhile examining premise systems for different representative closure operators. In Figure 1(b), $\{bf\}$ is required to be in its premise set $\mathcal{PS}$. It, too is a minimal non-closed set. The family $\{\{bc\}, \{cf\}, \{df\}, \{bf\}\}$ is a minimal premise system. We call such a premise system $\mathcal{PS}$ for which no proper subfamily is also a premise system, a **premise basis**.

Figure 1 denotes an antimatroid convex closure on an acyclic graph [23]. Figure 5 illustrates a general closure operator which is not anti-matroid in nature. The minimal non-closed sets $\{d\}, \{f\}, \{g\}, \{be\}, \{eh\}, \{ac\}, \{bh\}, \{ce\}, \{ah\}$ are elements of $\mathcal{PS}$, but so are others.

Lemma 5 *Let $\mathcal{PS}$ be any premise basis of $(\mathbf{U}, \varphi)$ and let Y be a pseudo-closed set such that $Y \in \mathcal{PS}$. If $Y \neq Y.\kappa$, then $(\mathcal{PS}{-}Y) \cup Y.\kappa$ is a premise basis of $(\mathbf{U}, \varphi)$.*

PROOF. Since $\mathcal{PS}$ is a premise basis, removal of Y must violate the second property of premise sets, so there exists Z not closed such that $Y \subseteq Z$ where $Y.\varphi \not\subseteq Z$. $Y.\kappa \subseteq Y$ and $Y.\kappa.\varphi = Y.\varphi$; so clearly $Y.\kappa$ preserves this property with respect to any such Z as well as Y. Thus, $\mathcal{PS}$ remains a premise basis. $\square$

Consider the non-closed sets $\{bd\}$ and $\{abd\}$. We claim one or the other must be in $\mathcal{PS}$, because otherwise only $\{d\} \in \mathcal{PS}, \{d\} \subseteq \{abd\}$ and $\{d\}.\varphi = \{ad\} \subseteq \{abd\}$ implying $\{abd\}$ is closed. Observe that $\{abd\}$ is pseudo-closed since $\{d\}$ is its only pseudo-closed proper subset.

Is one of $\{bd\}$ or $\{abd\}$ preferable to include in $\mathcal{PS}$? It is a toss up. In [7], Duquenne and Guiges demonstrated that for any closure system $(\mathbf{U}, \varphi)$, the pseudo-closed sets form a premise basis; but consequently we also have

Corollary 6 *Let $(\mathbf{U}, \varphi)$ be any closure system. There exists a premise basis $\mathcal{PS}$ composed only of generating sets, $Y.\kappa$.*

Similarly $\{abde\}$, $\{agh\}$, $\{adeh\}$ and $\{adgh\}$ are pseudo-closed and one can include either these or the corresponding generating sets $\{bde\}$, $\{ag\}$, $\{dh\}$ and $\{dg\}$ in $\mathcal{PS}$.

A closure space which has a unique premise basis is said to be **monotactic**. The closure space of Figure 1 is monotactic. Several other monotactic closure spaces are given in [16].

Notions of closure and of convexity abound in mathematics and computer science. Most often, these concepts are embedded in E^n, or some other continuous space. What we have shown is that these same concepts can be developed within discrete systems; that the resulting closure spaces are rich with structure; and that this structure can be profitably examined.

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