A Category of Discrete Closure Systems

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Abstract. Discrete systems such as sets, monoids, groups are familiar categories. The internal strucutre of the latter two is defined by an algebraic operator. In this paper we describe the internal structure of the base set by a closure operator. We illustrate the role of such closure in convex geometries and partially ordered sets and thus suggest the wide applicability of closure systems.
Next we develop the ideas of closed and complete functions over closure spaces. These can be used to establish criteria for asserting that "the closure of a functional image under f is equal to the functional image of the closure". Functions with these properties can be treated as categorical morphisms. Finally, the category **ClosureSys** of closure systems is shown to be cartesian closed.

1 Closure Systems

By a discrete system we mean a set of elements, points, or other phenomena which we will generically call our *universe*, denoted by $\mathbf{U}$. Individual points of $\mathbf{U}$ will be denoted by lower case letters: $a, b, ..., p, q, ... \in \mathbf{U}$. By $2^{\mathbf{U}}$, we mean the powerset on $\mathbf{U}$, or collection of all subsets of $\mathbf{U}$. Elements of $2^{\mathbf{U}}$ we will denote by upper case letters: S, T, X, Y, Z. A **closure system**, $\mathcal{C}$, is any collection of subsets $X, Y, \ldots Z \subseteq \mathbf{U}$, including $\mathbf{U}$ itself, which is closed under intersection. Subsets in $\mathcal{C}$ are said to be **closed**. If $\mathbf{U} = \{a, b, c, d, e\}$ then the collection of closed sets

$$\mathcal{C}_{\mathbf{1}} = \{\emptyset, \{a\}, \{b\}, \{ab\}, \{bd\}, \{abc\}, \{abd\}, \{abce\}, \{abcde\}, \{abcdef\}\} \tag{1}$$

is a closure system. We require the empty set $\emptyset$ to be included in all closure systems; although this will not be necessary until we define direct products in the category.

A closure system can equivalently be defined as $(\mathbf{U}, \varphi)$, where φ is a **closure operator** satisfying four axioms. For all $Y, Z \subseteq \mathbf{U}$,

C0: $\emptyset.\varphi = \emptyset$
C1: $Y \subseteq Y.\varphi$,
C2: $Y \subseteq Z$ implies $Y.\varphi \subset Z.\varphi$, and
C3: $Y.\varphi.\varphi = Y.\varphi$.

By C1, $\mathbf{U}$ itself must be closed. Here we are using a suffix operator notation, as we will throughout this paper. Read $Y.\varphi$ as "Y closure". A set Y is **closed** if $Y = Y.\varphi$. It is not hard to show that these two definitions of closure are equivalent.

A closure operator/system can satisfy other axioms depending on the mathematical discipline. A **topological closure** is closed under union, or

C4: $(Y \cup Z).\varphi = Y.\varphi \cup Z.\varphi$.

The closure operator of linear systems, often called the spanning operator, satisfies the Steinitz-MacLane **exchange axiom**

C5: if $p, q \notin Y.\varphi$ and $q \in (Y \cup \{p\}).\varphi$ then $p \in (Y \cup \{q\}).\varphi$.
Such closure systems are called **matroids**. Still other closure operators may satisfy an **anti-exchange axiom**

C6: if $p, q \notin Y.\varphi$ and $q \in (Y \cup \{p\}).\varphi$ then $p \notin (Y \cup \{q\}).\varphi$.
These closure operators, which include the geometric convex hull operator, are said to define **antimatroid** closure systems. An important antimatroid property is that:

Proposition 1. *Let φ be antimatroid. If $X.\varphi = Y.\varphi = Z$ closed, then $(X \cap Y).\varphi = Z$.*

Proof. If $X \cap Y = X$ or Y, the result is trivial. So there exists $pinX - Y$ and $q \in Y - X$. Now, suppose $(X \cap Y).\varphi \neq Z$, then $(X \cap Y \cup \{p\}).\varphi = Z = (X \cap Y \cup \{q\}).CL$ (if not let $X \cap Y \cup \{p\} = X'$ and $X \cap Y \cup \{q\} = Y'$ and repeat the argument) contradicting C6. □

Let $(\mathbf{U}, \varphi)$ be a closure system. Containment, $\subseteq$, forms a natural partial ordering on the closed subsets $\mathcal{C}_1$ shown in (1), and it is well known that the closed sets in $(\mathbf{U}, \varphi)$ so ordered form a lower semi-modular lattice.[1] Figure 1 illustrates the lattice of closed sets of the closure system $\mathcal{C}_1$ ordered by inclusion.

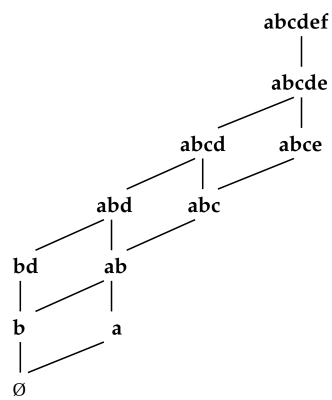

Fig. 1. The lattice of closed sets of $\mathcal{C}_1$

1.1 Relative Closure

In our development of closure space functions we will also use the following concept, which has the feeling and flavor of relative topologies. Let $(\mathbf{U}, \varphi)$ be any closure system and let $W \subseteq \mathbf{U}$. By the **relative closure φ with respect to W**, denoted $\varphi|_W$, we mean

$$Y.\varphi|_W = Y.\varphi \cap W, \quad \forall\, Y \subseteq W.$$

It is not hard to see that if φ is antimatroid, then $\varphi|_W$ is as well.

As shown below, the restriction of a closed set will always be closed. The ability to infer that Y is closed wrt. φ when its restriction is closed wrt. $\varphi|_W$ is of more interest. The following proposition from [12] gives two sufficient conditions. Neither is necessary.

[1] This lower semimodularity of closed subsets partially ordered by inclusion has been repeatedly discovered by many authors. See Monjardet [10] for an interesting summary.

Proposition 2. *In any closure system* $(\mathbf{U}, \varphi)$

(a) X *closed wrt.* φ *implies* $X \cap W$ *is closed wrt.* $\varphi|_W$.

(b) *If* W *is closed wrt.* φ *and* $X \subseteq W$, *then* X *closed wrt.* $\varphi|_W$ *implies* X *is closed wrt.* φ.

(c) *If* X *is closed wrt.* $\varphi|_W$ *and* $(X.\varphi - X) \cap (\mathbf{U} - W) = \emptyset$, *then* X *closed wrt.* φ.

Closure systems are common in discrete applications. One encounters transitive closure in many guises. Three important types of closure system are 1) convex geometries, 2) poset closure, and 3) Galois closure of relations. Since Galois closure need not be antimatroid [5], we will ignore it in this paper. The other two are worth examining in a bit more detail because they can provide many motivating examples.

1.2 Convex Geometries

By a *convex geometry* we shall mean a collection of discrete points in an n-dimensional Euclidean space. For the illustrative purposes of this paper, we always assume a 2-space. The closure operator is the convex hull operator. A much more complete treatment can be found in [3, 4, 6]. There the closed sets are called *alignments*.

A convex hull closure operator provides the quintessential example of antimatroid closure. Consider Figure 2. The point q is in the convex hull of $Y \cup \{p\}$, consequently p cannot possibly be in

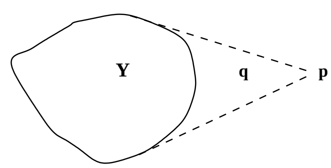

Fig. 2. Illustration of antimatroid closure

the convex hull of $Y \cup \{q\}$.

1.3 Posets as Closure Systems

Suppose $\mathbf{U}$ is a poset $(P, \leq)$ such as Figure 3, where $a < c < e$, *etc.* Or, equivalently, we may think of Figure 3 as an acyclic graph $G = (P, E)$. Two familiar closure operators on P are

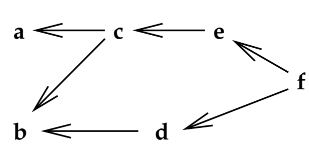

Fig. 3. A 6 point poset

$Y.{\downarrow} = \{\, x \mid \exists y \in Y, x \leq y \,\}$ and

$Y.\!\uparrow = \{ z \mid \exists y \in Y, y \leq z \}$.

"Downset closure", $\downarrow$, on Figure 3 yields the closure system $\mathcal{C}_1$ introduced in Section 1 and the closure lattice of Figure 1.[2] $Y.\!\downarrow$ is sometimes called an ideal, and $Y.\!\uparrow$ a filter. In this paper we will emphasize the **downset** closure, $\downarrow$, on posets; although others are possible.

A potentially more interesting closure operator on posets is the **convex closure** operator, denoted by φ_C, defined $Y.\varphi_C = \{ x \mid \exists y_1, y_2 \in Y, y_1 \leq x \leq y_2\}$. Some authors call these closed sets "causal sets [1]. The lattice of closed sets generated by φ_C operating on the 5 point poset of Figure 3 is shown in Figure 4. Observe how many more closed subsets of $P = \{a, b, c, d, e\}$ are created by

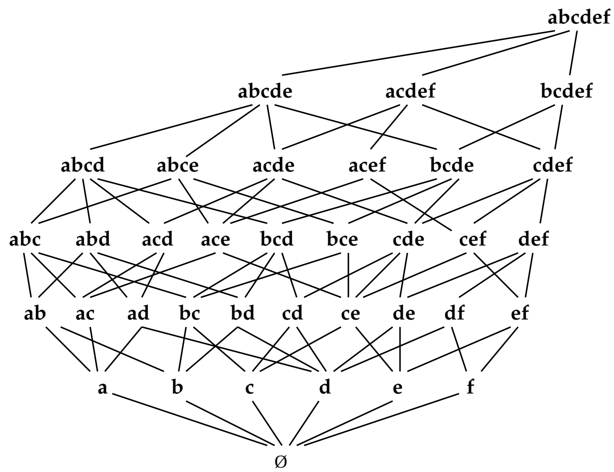

Fig. 4. The lattice of closed sets formed by convex closure, φ_C on the poset of Figure 3.

φ_C than by $\downarrow$. It is easy to show that $\varphi_C = \downarrow \cap \uparrow$. We call $\downarrow, \uparrow$, and φ_C collectively **path closures**.

We now add two important concepts to our development of closure objects; they are the notion of *generators* and an ordering of *all* subsets of $\mathbf{U}$, not just the closed subsets.

1.4 Generators

Let Y be a closed set. A subset $X \subseteq Y$ is said to generate Y if $X.\varphi = Y$. By C4, every closed set Z generates itself. But, that tends to be uninteresting. We say a generator Y of Z is **non-trivial** if $Y \subset Z$. In fact, we are really only interested in *minimal* non-trivial generators. And, because φ is antimatroid know these minimal elements are unique (as we show immediately below). More specifically, if X is the minimal generating set, it is called the **generator** of Y, and denoted by $Y.\gamma$.[3] When there is only a single generating set for any closed set Y we say that $(\mathbf{U}, \varphi)$ is **uniquely generated**. The three path closures, $\downarrow, \uparrow, \varphi_C$, on posets are all uniquely generated.

Proposition 3. *A closure system* $(\mathbf{U}, \varphi)$ *is antimatroid if and only if it is uniquely generated.*

[2] Since we draw partial orders from left to right, perhaps this should be called a "left set" ordering!

[3] When a closed set may have more than one generator, the collection of all minimal generating sets we denote by $Y.\Gamma = \{Y.\gamma_1, \ldots, Y.\gamma_n\}$ [5].

Proof. If $(\mathbf{U}, \varphi)$ is not antimatroid then there exists some closed set $Y.\varphi$ with $p, q \notin Y.\varphi$ such that $p \in (Y \cup q).\varphi$ and $q \in (p \cup Y).\varphi$. Then p and q are members of distinct generators of $(Y \cup p).\varphi = (Y \cup q).\varphi$. The converse is similarly shown.
This proposition can also be treated as a corollary of Prop. 1. $\Box$

If $(\mathbf{U}, \varphi)$ is antimatroid, then the collection of all sets X_i with the same closure $Y = X_i.\varphi$ constitute a Boolean lattice with $Y.\gamma \subseteq X_i \subseteq Y$. In particular, if $X_i, \ldots X_k$ generate Y then $X_i \cap \ldots \cap X_k$ also generates Y.

1.5 Partially Ordering a Power Set with respect to Closure

In the preceding sections, we partially ordered only the closed sets by containment. Now we seek to partially order *all* the subsets of $\mathbf{U}$, simple containment no longer seems quite so natural. In particular, containment does not involve the closure operator. To partially order $2^{\mathbf{U}}$ with respect to the closure φ, we say

$$X \sqsubseteq Z \quad \text{if and only if} \quad Z \cap X.\varphi \subseteq X \subseteq Z.\varphi \tag{2}$$

for all $X, Z \subseteq \mathbf{U}$, It is not hard to show that $\sqsubseteq$ really is an ordering; only transitivity takes a bit of thought [12]. The value of this ordering we will see as we progress.

First, we observe that if X and Z are closed in $(\mathbf{U}, \varphi)$ then $X \sqsubseteq Z$ if and only if $X \subseteq Z$, so on the closed sets it coincides with the subset partial ordering. Second, if X_i and X_k generate Z then $X \sqsubseteq X_k$ if and only if $X_k \subseteq X_i$. Third, it can be shown [12] that if $\mathbf{U}$ is finite and φ is antimatroid then $(\mathbf{U}, \sqsubseteq)$ defines a complete lattice, $\mathcal{L}$, called its **closure lattice** [12]. Again, we should properly use the notation $(2^{\mathbf{U}}, \sqsubseteq)$ since $\sqsubseteq$ is an ordering of the powerset, but this abuse of notation emphasizes the role of the base set $\mathbf{U}$. Figure 5 illustrates the closure lattice generated by the downset operator $\downarrow$ on the poset of Figure 3. We had seen Figure 4 as its lattice of closed subsets. In Figures 1 and 5, the top of the lattice has been pushed to the right to more easily draw the Boolean generator sets in an upper left to lower right orientation. The regularity of structure suggested by this figure really exists, *c.f.* [12]. The sublattice of closed sets in this figure is denoted by set labels that are joined by solid lines that are generally inclined from the upper right to the lower left.

The generators, d, ad, c, cd, e, de and f, are connected to the corresponding closed sets that they generate by dashed lines generally inclined from the upper left to the lower right. Observe that each of the lattice intervals $[Y.\varphi, Y.\gamma]$ is a boolean sublattice. In the case of the 32 subsets comprising the boolean sublattice $[abcdef, f]$ and the 8 subsets comprising $[abcde, de]$ and $[abce, e]$, we only indicate a few of their their constituent elements and a dashed outline.

The dash-dot lines denote a few of the covering relationships between non-closed elements in different boolean intervals. These covering relationships, which correspond to the ordering $X \sqsubseteq Z$, echo those of the closed subgraph sublattice. This pattern will be seen in all closure spaces where if X_1 is a generator of $Z_1 \sqsubseteq Z_2$, then there exists a generator X_2 of Z_2 such that $X_2 - X_1 = Z_2 - Z_1$ [12].

Finally, we observe that this closure is T_0, that is union preserving (C4) and antimatroid (C6) so this order $\sqsubseteq$ is precisely the *specialization order*, $\sqsubseteq_s$ described in [8].

When we used convex closure, φ_C, on the poset of Figure 3 we had many more closed subsets than were in Figure 5. Consequently, we have many fewer "non-trivial" generators as shown in Figure 6. Again, dashed lines demarcate subsets with the same closure. Observe that convex closure on any

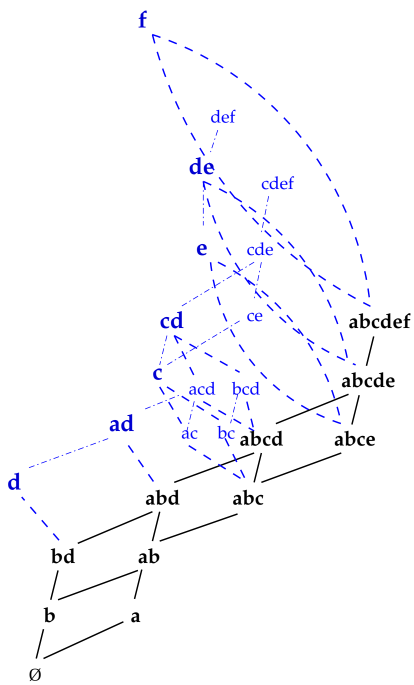

Fig. 5. A Closure Lattice, $\mathcal{L}_\varphi$ with the closed sets of Figure 4.

poset is antimatroid, that $|\,Y.\gamma\,| \geq 2$ and that any finite set closed with respect to downset $\downarrow$ or upset $\uparrow$ will be closed with respect to φ_C.

The following proposition establishes that with posets the closure space ordering is conformant to the poset order. This, and the following 3 results, come from [12].

Proposition 4. *Let $(P, \downarrow)$ be a poset. Then $x \leq y$ if and only if $x \sqsubseteq y$.*

Proof. $x \leq y$ if and only if $x \in y.\!\downarrow$. And readily $\{y\} \cap \{x\}.\!\downarrow \neq \emptyset$ if and only if $x = y$, so $\{y\} \cap \{x\}.\!\downarrow \subseteq \{x\} \subseteq \{y\}.\!\downarrow$. □

There are many interesting relationships between subset inclusion $\subseteq$ and the closure space (lattice) ordering $\sqsubseteq$ as determined by the closure operator φ.

Proposition 5. *If $P \cap X = \emptyset$ then*
(a) $X \sqsubseteq X \cup P$ if and only if $P \cap X.\varphi = \emptyset$,
(b) $X \cup P \sqsubseteq X$ if and only if $P \subseteq X.\varphi$.

Proof. $X \sqsubseteq X \cup P$ iff $(X \cup P) \cap X.\varphi \subseteq X \subseteq (X \cup P).\varphi$, where the second containment always holds, but the first is true iff $P \cap X.\varphi = \emptyset$.
$X \cup P \sqsubseteq X$ iff $X \cup (X \cup P).\varphi \subseteq X \cup P \subseteq X.\varphi$, where readily we need $P \subseteq X.\varphi$ for the second containment to be true. □

Proposition 6. *Let $X \sqsubseteq Z$. If Z is closed then X is closed.*

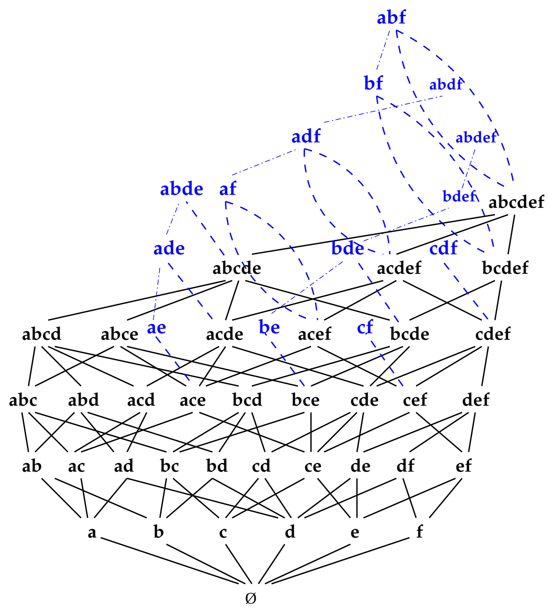

Fig. 6. Generators of the convex closure φ_C of Figure 4

Proof. $Z \cap X.\varphi \subseteq X \subseteq Z = Z.\varphi$. Thus $X \subset Z$ and $X \subseteq X.\varphi$ so $X \subseteq X \cap X.\varphi$ which as the intersection of closed sets must be closed. Since $X.\varphi$ is the smallest closed set containing X, $X = X \cap X.\varphi = X.\varphi$. □

Sometimes the contrapositive, that is if $X \sqsubseteq Z$ and X is not closed then Z cannot be closed, is a more useful formulation.

In a discrete closure space, covering relationships take on a special importance, just as the existence of individual edges can be important in graph theory even though the transitive relationship is the path relation. Recall that a lattice element (subset) Z **covers** the element X in $\mathcal{L}$ if $X \sqsubseteq Z$ and if $X \sqsubseteq Y \sqsubseteq Z$ implies $X = Y$.

Proposition 7. *If φ is antimatroid and Z is closed, then for all $p_k \in Z.\gamma$, Z covers Y_k if and only if $Y_k = Z - p_k$.*

Proof. Let $Y_k = (Z.\gamma - p_k).\varphi \subset Z$. Readily $p_k \notin Y_k$, hence $Y_k \subseteq Z \not{p}_k \subset Z$. Since $Z \not{p}_k$ does not generate Z, Z covers $X - p_k$ in $\mathcal{L}$, so by Prop. 6, $Y_k = Z - p_k$ is closed.
The converse is similar. □

It is worth verifying this relationship in Figure 5, as well as all other antimatroid closure spaces. A more general relation between the generators of a closed set Z and the closed sets that it covers can be found in [5].

1.6 Generators in Convex Geometries and Concept Lattices

A closed convex hull is generated by elements of its boundary. The minimal generating set is precisely the vertices of the boundary. Since, in a discrete geometry, these must be unique, convex hull closure is antimatroid. The efficiency of the Simplex method for solving linear inequalities [2] is based on this principle. In Figure 7 there is a small 6 point geometry. Every singleton point or doubleton pair is a trivial closed set. This is reflected in the "busy" structure of the closure lattice to the right. The non-trivial generating sets are $\{abcd, abcf, abdf,\ acde, bcde, abc,\ abd, abf, acd,\ bcd, cde\}$. In [3, 4,

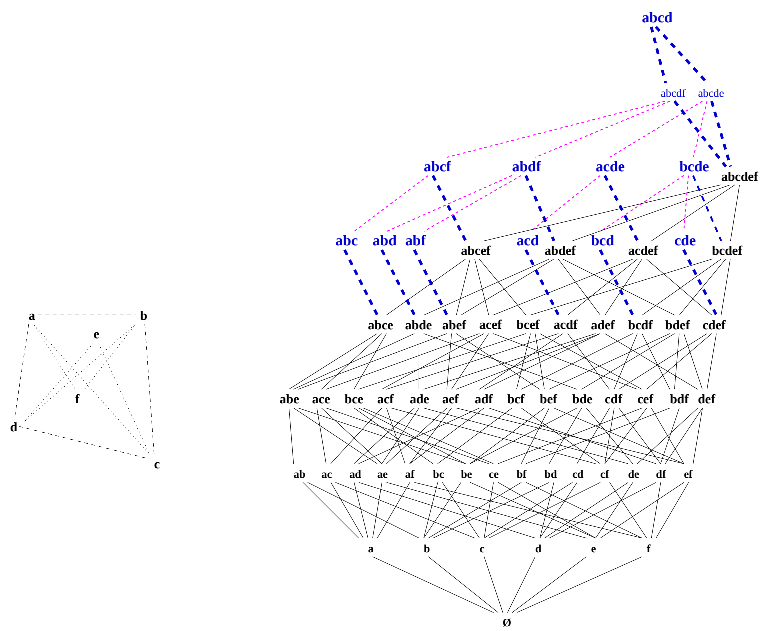

Fig. 7. A small 6 point convex geometry

6] they are called *extreme* points. The set of points $abcd$ generate the entire space $(\mathbf{U}, \varphi)$. Following Proposition 7, deletion of any generating (extreme) point, say b, from $Z = abcdef$ must create a new closed set $Y = acdef$. In [9], this repeated deletion of generating/extreme points is called "shelling" where they use it to define antimatroid behavior.

2 Functions on Closure Systems

A function f mapping a closure system $\mathcal{C}$ into another $\mathcal{C}'$ is a function taking the power set of $\mathbf{U}$ into that of $\mathbf{U}'$. To simplify notation, we denote these functions by $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ instead of the more correct $2^{\mathbf{U}} \xrightarrow{f} 2^{\mathbf{U}'}$.

Functions, or morphisms, taking $2^{\mathbf{U}}$ to $2^{\mathbf{U}'}$ can be arbitrarily complex. To reduce the complexity somewhat, we consider only **monotone** functions f for which $X \subseteq Y$ in $\mathbf{U}$ implies $X.f \subseteq Y.f$ in $\mathbf{U}'$.

Let $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ be a function between the power sets of $\mathbf{U}$ and $\mathbf{U}'$. Now, consider $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ as a function between the closure systems $(\mathbf{U}, \varphi)$ and $(\mathbf{U}', \varphi')$. Does the function f preserve the closure structure of $\mathbf{U}$ in $\mathbf{U}'$? More particularly, does the following diagram commute, that is $\forall X \subseteq \mathbf{U}$ does $X.f.\varphi' = X.\varphi.f$?

$$
\begin{tikzcd}
\mathbf{U} \arrow[r, "f"] \arrow[d, "\varphi"] & \mathbf{U}' \arrow[d, "\varphi'"] \\
(\mathbf{U}, \varphi) \arrow[r, "f"] & (\mathbf{U}', \varphi')
\end{tikzcd}
$$

Fig. 8. f regarded as a closure system transformation

2.1 Complete and Closed Functions

A function $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ is said to be **complete** if $\forall X \in \mathbf{U}$, $X.f$ closed in $(\mathbf{U}', \varphi')$ implies $X.\varphi.f = X.f$.

In a discrete space, "completeness" has some of the characteristics more commonly associated with "continuity". In particular, it provides an analog to the notion that "the inverse image of closed sets is closed". To get a sense of why we might want a function to be "complete" consider the simple function f which maps a linear order on the 3 points, a, b, and c shown in Figure 9 onto a linear order on just 2 points a', c', where $a.f = a', c.f = c'$ and $b.f$ can be either a' or c'. This f is clearly

$$\mathbf{a} \longleftarrow \mathbf{b} \longleftarrow \mathbf{c} \quad \xRightarrow{\text{f}} \quad \mathbf{a'} \longleftarrow \mathbf{c'}$$

Fig. 9. A "complete" function

"order preserving" and could be regarded as an epitome of a "well-behaved" discrete function. The subset $a'c'$ is closed with respect to any of the closure operators that we normally associate with such partial orders, while $ac \in \{a'c'\}.f^{-1}$ is closed with respect to none of them. Clearly, we can't require every pre-image set to be closed; but by requiring that whenever the image is closed, the closure of a pre-image set must also map onto this image set, we believe we have captured an essential property. The next sequence of propositions provide additional confirmation of this.

Properly, $Y'.f^{-1}$ is a collection of sets that map onto Y'. However, it is more convenient to let $Y'.f^{-1}$ denote $\bigcup_Y \{\, Y \;:\; Y.f = Y' \,\}$.

Proposition 8. *Let $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ be complete. If Y' is closed in $\mathbf{U}'$, then $Y'.f^{-1}$ is closed in $\mathbf{U}$.*

Proof. Since $Y'.f^{-1}.f = Y'$, which is closed, by completeness we have $Y'.f^{-1}.\varphi.f = Y'$ or $Y'.f^{-1}.\varphi = Y'.f^{-1}$. $\square$

Thus the inverse image of a closed set is indeed closed.

Proposition 9. *A function* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is complete if and only if* $\forall X \subseteq \mathbf{U},\ X.\varphi.f \subseteq X.f.\varphi'$.

Proof. $X.f \subseteq X.f.\varphi'$. Let $Y' = X.f.\varphi'$, so Y' is closed. Let $Y = Y'.f^{-1}$ Since $X \subseteq Y$, by completeness, $X.\varphi.f \subseteq Y.f = X.f.\varphi'$.
Conversely, let $X.f$ be closed. Then $X.\varphi.f \subseteq X.f.\varphi' = X.f$. Now $X \subseteq X.\varphi$, so $X.f \subseteq X.\varphi.f$ and equality holds. $X.f.\varphi' = X.f$ and f is complete. □

We will say a function $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ is **closed** if f takes closed sets in $(\mathbf{U}, \varphi)$ onto closed sets in $(\mathbf{U}', \varphi')$

Proposition 10. *A function* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is closed if and only if* $\forall X \subseteq \mathbf{U},\ X.f.\varphi' \subseteq X.\varphi.f$.

Proof. Let f be closed. $X \subseteq X.\varphi$ implies $X.f \subseteq X.\varphi.f$. But, because $X.\varphi$ is closed and f is closed, $X.f.\varphi' \subseteq X.\varphi.f$
Conversely, let X be closed in $(\mathbf{U}, \varphi)$. $X.f.\varphi' \subseteq X.\varphi.f = X.f$. But, readily $X.f \subseteq X.f.\varphi'$ so equality holds. □

Theorem 1. *A function* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is closed and complete if and only if for all* $X \subseteq \mathbf{U}$, $X.\varphi.f = X.f.\varphi'$.

Proof. Propositions 10 and 9. □

Theorem 1 provides necessary and sufficient conditions on a monotone function f so that f commutes with the closure operator φ as in Figure 8.

Proposition 11. *Let* $\mathbf{U} \xrightarrow{f} \mathbf{U}', \mathbf{U}' \xrightarrow{g} \mathbf{U}''$ *be complete (closed) then* $\mathbf{U} \xrightarrow{f \cdot g} \mathbf{U}''$ *is complete (closed).*

Proof. That the composition of closed functions is closed is trivial.
Let f and g be complete and let $Y.(f \cdot g) = Y.f.g = Y'' \in \mathbf{U}''$ be closed. We must show that $Y.\varphi.(f \cdot g) = Y''$. Since g is complete, $(Y.f.\varphi').g = Y''$ and f complete implies first that $Y.\varphi.f \subseteq Y.f.\varphi'$ and then that $Y.\varphi.f = Y.f.\varphi'$. Thus, $Y.\varphi.f.g = Y.f.\varphi'.g = Y''$. □

2.2 Homomorphisms of Posets

Just as one normally defines a closure system by an operator φ on the base set $\mathbf{U}$ rather than by enumerating the closed sets, so too one commonly defines functions between the power sets $2^{\mathbf{U}}$ and $2^{\mathbf{U}'}$ by means of a point function $f : \mathbf{U} \to \mathbf{U}'$. We then lift f is the usual way to $X.f^{+} = \{y' \in \mathbf{U}' \mid \exists x \in X, x.f = y'\}$. Similarly, $Y'.f^{-} = \{x \in \mathbf{U} \mid \exists y' \in Y', x.f = y'\}$. Observe that f^{-} is identical to f^{-1} we have been already using. The lifting notation f^{+} and f^{-} seems to be due to G.Q. Zhang [14, 15].

A lifted function f must be monotone, as described at the beginning of Section 2. They provide an excellent source of examples and counter examples. In this section we examine homomorphisms on posets.

Recall that $f : P \to P'$ is a **homomorphism** of the poset P into P' if $x \leq y$ in P implies $x.f \leq y.f$ in P'. Since an anti-chain can be the pre-image of any ordered set under a homomorphism, it would be an exaggeration to call them "structure preserving". Often one wants the structure of the domain to have some resemblence to its image. For this reason some authors prefer "strong homomorphisms" where a homomorphism is called **strong** if in addition $x' \leq y'$ in P' implies $\exists x, y \in P$ such that $x.f = x', y.f = y'$ and $x \leq y$.

Proposition 12. *A function $f : (P, \leq) \rightarrow (P', \leq')$ is a homomorphism if and only if the lifted function $(P, \varphi) \xrightarrow{f^+} (P', \varphi')$ is complete with respect to a path closure.*[4]

Proof. Let f be a homomorphism and let $Y \subseteq P$. For all $y \in Y$, if $x \leq y$ then $x.f \leq y.f$ So $Y.\!\downarrow\!.f^+ \subseteq Y.f^+.\!\downarrow$ and by Prop. 9 f^+ is complete.
Conversely, assume f^+ is complete and that $x \leq y$. Then $x \in y.\!\downarrow$ and $x.\!\downarrow \subseteq y.\!\downarrow$. Again by Prop. 9, $\{x\}.\!\downarrow\!.f^+ \subseteq \{y\}.f^+.\!\downarrow$ or $x.f \leq y.f$.
The proof when the closure is φ_C or $\uparrow$ is similar. □

In an early application of this principle [11] it was shown that if $f : G \rightarrow G'$ was any graph homomorphism, the G' would be acyclic if and only if the inverse image of every point in G' was a convex (in the sense of φ_C of Section 1.3) set in G.

But, poset homomorphisms need not be closed. In Figure 10, $\{a, b, d\}$ is closed (assuming $\downarrow$

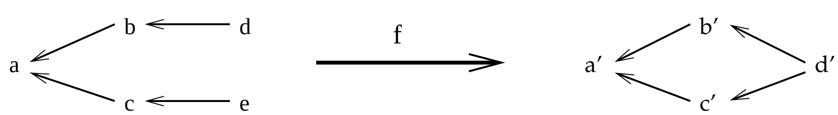

Fig. 10. A homomorphism f that is not closed.

closure), but $\{a', b'd'\}$ is not. Even strong homomorphisms need not be closed, as shown by Figure 11, because $\{b_2\}$ is closed but $\{b_2\}.g = \{b'\}$ is not. However, we can show

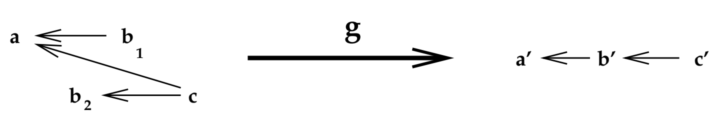

Fig. 11. A strong homomorphism g that is not closed.

Proposition 13. *Let $g : (P, \leq) \rightarrow (P', \leq)$. If $(P, \varphi) \xrightarrow{g^+} (P, \varphi')$ is closed with respect to a path closure then g is a (strong) homomorphism.*

Proof. Assume that $\downarrow$ is the closure φ. Let $x \leq y$ or $x \in y.\!\downarrow$, so $x.g^+ \in y.\!\downarrow\!.g^+$. Since $y.\!\downarrow$ is closed, $y.\!\downarrow\!.g^+$ is closed, so $x.g^+ \in y.\!\downarrow\!.g^+.\!\downarrow$ or $x.g \leq y.g$. g is a homomorphism.

[4] Note that we use the traditional notation $f : \mathbf{U} \rightarrow \mathbf{U}'$ for point functions on $\mathbf{U}$, and $\mathbf{U} \xrightarrow{f^+} \mathbf{U}'$ for functions between the power sets.

Now let $x' \leq y'$ in P', and let $\{y'\}.g^{-1} = Y$. $Y.g^{+} = \{y'\}$ so $x' \in Y.g^{+}.\!\downarrow \subseteq Y.\!\downarrow\!.g^{+}$ implying there exists $y_k \in Y$ and $x_i \in Y.\!\downarrow$ so $x_i \leq y_k$ where $x_i.g = x'$ and by definition of Y, $y_k = y'$. So g^{+} is strong. $\square$

Combining Prop. 12 with Prop. 13 we readily have that if g^{+} is closed, then g is a homomorphism so g^{+} must also be complete. But, only in this special case where we have lifted functions over poset closures need this be true. It is not a general implication.

2.3 Intersection of Functions, $f \cap g$

One can define the operators $+, -, \times$ and $\div$ on numeric functions in a pointwise fashion; that is, $f + g$ is defined $\forall x$ to be $x.f + x.g$. Since the domain of our functions is a power set, we define the intersection, $\cap$, or meet, of two functions on a closure system in a "set-wise" fashion as, $X.(f \cap g) = X.f \cap X.g$. Earlier we had observed that $\varphi_C = \downarrow \cap \uparrow$.

Our next step is to show that the meet of closed (complete) functions is closed (complete).

Proposition 14. *If* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *and* $(\mathbf{U}, \varphi) \xrightarrow{g} (\mathbf{U}', \varphi')$ *are closed functions then* $(\mathbf{U}, \varphi) \xrightarrow{f \cap g} (\mathbf{U}', \varphi')$ *is a closed function.*

Proof. Let Y be closed. Since f and g are closed functions, $Y.f$ and $Y.g$ are closed sets in $\mathbf{U}'$ and since $(\mathbf{U}', \varphi')$ is a closure system, $Y.(f \cap g) = Y.f \cap Y.g$ is closed. $\square$

Proposition 15. *If* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *and* $(\mathbf{U}, \varphi) \xrightarrow{g} (\mathbf{U}', \varphi')$ *are complete functions then* $(\mathbf{U}, \varphi) \xrightarrow{f \cap g} (\mathbf{U}', \varphi')$ *is a complete function.*

Proof. Suppose $Y.(f \cap g)$ is closed in $\mathbf{U}'$. Either $Y.f.\varphi' = Y.g.\varphi'$ or not. If $Y.f.\varphi' \neq Y.g.\varphi'$ then by Prop. 14, $Y.(f \cap g) = Y.f.\varphi' \cap Y.g.\varphi'$. By completeness of f and g, $Y.\varphi.f = Y.f.\varphi'$ and $Y.\varphi.g = Y.g.\varphi'$. So $Y.\varphi.(f \cap g) = Y.\varphi.f.\varphi' \cap Y.\varphi.g.\varphi' = X.f.\varphi' \cap Y.g.\varphi' = Y.(f \cap g)$.
If $Y.f.\varphi' = Y.g.\varphi'$ it's a bit more direct, $Y.(f \cap g) = Y.(f \cap g).\varphi' = (Y.f \cap Y.g).\varphi' = Y.f.\varphi'$. $\square$

Let $\mathcal{F}$ be a family of functions $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$. We say the family $\mathcal{F}$ is **closed** if for all $f, g \in \mathcal{F}, f \cap g \in \mathcal{F}$. This is a standard definition using "closure under intersection". (There are entirely too many common mathematical meanings of the term "closed" and "closure". A set is "closed" if ...; a function is "closed" if ...; an operator is "closed" if ... We count on context to disambiguate them.) We now define a closure operator φ in the "usual way".

Let $[\mathbf{U} \to \mathbf{U}']$ denote the universe of all functions $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$, or all such functions satisfying specific properties, such as completeness or closure. Let $\mathcal{F}$ be any subset of $[\mathbf{U} \to \mathbf{U}']$ then $\mathcal{F}.\varphi = \{\, g \in [\mathbf{U} \to \mathbf{U}'] \mid g = \bigcap_i f_i, \text{ where } f_i \in \mathcal{F} \,\}$.

Proposition 16. *Let* $\mathcal{C}_1, \mathcal{C}_2$ *be closure systems. The family of all closed, complete functions* $[\mathbf{U}_1 \xrightarrow{f} \mathbf{U}_2]$ *is a closure system.*

Proof. It is sufficient to just observe that $f \cap g \in [\mathcal{C}_1 \xrightarrow{f} \mathcal{C}_2]$. Alternatively, one can define the closure operator φ as above. It is not hard to show that the 3 closure axioms, C1, C2, and C3, are satisfied. $\square$

3 The Category ClosureSys

Because we have composition (Prop. 11), the collection of all closure systems together with all complete, closed functions as its morphisms can be easily shown to constitute a category, say **ClosureSys**.

Let $\mathcal{C} \in$ **ClosureSys**. The empty set, $\emptyset$, is the unique **terminal object** of $\mathcal{C}$ because the morphism f defined $X.f = \emptyset, \forall X \in \mathcal{C}$ is unique. Observe that in the category **Set**, $\emptyset$ is the initial object and individual elements are terminal objects [13]. The difference is that in a closure system $\emptyset$ is literally a singleton element, not shorthand for the "absence" of any elements.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be closure systems over the base sets $\mathbf{U}_1$ and $\mathbf{U}_2$. (For simplicity, we assume that $\mathbf{U}_1$ and $\mathbf{U}_2$ are disjoint, else we employ uniquely named copies.) We let the disjoint union $\mathbf{U}_1 \uplus \mathbf{U}_2$ be the base set of the **direct product** $\mathcal{C}_1 \times \mathcal{C}_2$, and say that for all closed $X_i \in \mathcal{C}_1$, $Y_j \in \mathcal{C}_2$, $X_i \uplus Y_j = Z_k$ is closed in $\mathcal{C}_1 \times \mathcal{C}_2$.

Proposition 17. *The closure system* $(\mathbf{U}_1, \varphi_1) \times (\mathbf{U}_2, \varphi_2) = 2^{\mathbf{U}_1 \uplus \mathbf{U}_2}$ *is a categorical direct product.*

Proof. First, $\mathcal{C}_1 \times \mathcal{C}_2$ so defined really is a closure system because if Z_1, Z_2 are closed sets in $\mathcal{C}_1 \times \mathcal{C}_2$, $Z_1 \cap Z_2 = (X_1 \cap X_2) \uplus (Y_1 \cap Y_2) = (X_1 \uplus Y_1) \cap (X_2 \uplus Y_2) \in \mathcal{C}_1 \times \mathcal{C}_2$. We can now define **projection** morphisms $\mathcal{C}_1 \times \mathcal{C}_2 \xrightarrow{\pi_1} \mathcal{C}_1$, $\mathcal{C}_1 \times \mathcal{C}_2 \xrightarrow{\pi_2} \mathcal{C}_2$, by $Z.\pi_i = Z \cap \mathbf{U}_i$. By Proposition 2, $Z \cap \mathbf{U}_i$ is closed. And if $\emptyset \in \mathcal{C}_i$, then π_i is a surjective morphism onto $\mathcal{C}_i$ because $Z_1 \cap \emptyset_2 = Z_1 \in \mathcal{C}_1$ and $\emptyset_1 \cap Z_2 = Z_2 \in \mathcal{C}_2$.
Finally, if $\mathcal{C}$ is any closure system with morphisms $\mathcal{C} \xrightarrow{f} \mathcal{C}_1$ and $\mathcal{C} \xrightarrow{g} \mathcal{C}_2$, we let h be the morphism, $\forall Z \in \mathcal{C}$ $Z.h = Z.f \uplus Z.g \in \mathcal{C}_1 \times \mathcal{C}_2$. Thus, $f = h \cdot \pi_1$ and $g = h \cdot \pi_2$. □

Consequently, **ClosureSys** is a category with direct products and unique terminal objects.

It is interesting to note that products in closure systems "feel" very much like the disjoint unions (or co-products) in **Set** or **Matroid**. However, it cannot be a co-product in **ClosureSys**, even though defining injections is trivial, because the arrow h needed to make the appropriate diagram commute need not be unique.

Proposition 18. *Let* $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ *be closure systems, and let* $\mathcal{C}_1 \times \mathcal{C}_2 \xrightarrow{f} \mathcal{C}_3$*. (I.e. f is a function of two variables.) Then f is closed (complete) if and only if f is closed (complete) on each of* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*.*

Proof. Let Z be closed in $\mathcal{C}_1$. Then $Z.f = (Z \cup \emptyset_2).f = (Z \times \emptyset_2).f$ which is closed in $\mathcal{C}_3$.
Completion and the converse are demonstrated analogously. □

The power of category theory appears to arise from situations where the morphisms on the category can be also regarded as objects in the category. Examples are linear spaces (matroids) where the linear operators can be treated as linear spaces themselves; or algebraic groups where the homomorphisms also exhibit group properties. We have the same phenomena with directed complete partial orders, or *dcpo*'s, where it is shown [7] that the set of (Scott)-continuous functions $D \xrightarrow{f} E$, where D and E are *dcpo*'s is itself a *dcpo*. This appears to be the essential step in showing that a category is cartesian closed.

With Proposition 16, which establishes that a collection of closed, complete functions can be regarded as a closure system, we have provided a basic motivation for the possibility of treating sets of morphisms as objects in **ClosureSys**. By making a slight notational change, we can make this embedding more evident.

All objects in **ClosureSys** will be represented by collections of ordered pairs (X, Y) of sets with $X \in 2^{\mathbf{U}}, Y \in 2^{\mathbf{U}'}$ where possibly $\mathbf{U} = \mathbf{U}'$. We say $< \mathbf{U}, \mathbf{U}' >$ is the **type** of the element in **ClosureSys**. The category **ClosureSys** can then be defined by a single axiom:

A1: Let $(X_1, Y_1), (X_2, Y_2) \in$ **ClosureSys** both be of type $< \mathbf{U}, \mathbf{U}' >$,
if $X_1 = X_2$ then $(X_1, Y_1 \cap Y_2) \in$ **ClosureSys**.

Now, we consider the consequences of this notational change. Let $\mathcal{C}$ denote some closure system over $\mathbf{U}$, then $\mathcal{C}$ consists of a collection of $2^{|\mathbf{U}|}$ elements of type $< \mathbf{U}, \mathbf{U} >$. If Z is a closed set of $\mathbf{U}$, that is $Z \in \mathcal{C}$, we represent Z by the pair $(\mathbf{U}, Z)$. If Y is not closed, it is represented by $(Y.\varphi.\gamma, Y)$, that is the left side is the generator of Y's closure. We can verify that these pairs satisfy the categorical axiom, A1. If Z_1 and Z_2 are closed in $\mathbf{U}$, and so denoted $(\mathbf{U}, Z_1), (\mathbf{U}, Z_2)$, they force $(\mathbf{U}, Z_1 \cap Z_2)$ to be closed as well. This, of course, is the basic property of all closure systems. If Y_1, Y_2, denoted by $(X, Y_1), (X, Y_2)$, are not closed, then Y_1 and Y_2 are both in the boolean interval $[X.\varphi, X]$ generated by X, and because φ is antimatroid by Proposition 1 $Y_1 \cap Y_2 \in [X.\varphi, X]$ as well, and so $(X, Y_1 \cap Y_2) \in$ **ClosureSys** as required by A1.

Now, let (X, Y) have type $< \mathbf{U}, \mathbf{U}' >, \mathbf{U} \neq \mathbf{U}'$. Then $(X, Y) \in 2^{\mathbf{U}} \xrightarrow{f} 2^{\mathbf{U}'}$. If there exists another element (X, Z) with type $< \mathbf{U}, \mathbf{U}' >$ then $(X, Z) \in 2^{\mathbf{U}} \xrightarrow{g} 2^{\mathbf{U}'}$ and by the axiom A1, $(X, Y \cap Z) \in 2^{\mathbf{U}} \xrightarrow{f \cap g} 2^{\mathbf{U}'}$ in **ClosureSys**.

Lastly, for every $X \in 2^{\mathbf{U}}$ we must throw into **ClosureSys** the pair (X, X) of type $< \mathbf{U}, \mathbf{U} >$ corresponding to the morphism $id_{\mathbf{U}}$. It is easy to verify that this addition completely conforms to the categorical axiom.

Consequently the category **ClosureSys** is a massive collection of typed ordered pairs. Those of type $< \mathbf{U}, \mathbf{U} >$ denote a closure system over $\mathbf{U}$, or else the identity morphism $id_{\mathbf{U}}$.[5] Different closure systems over the same base set $\mathbf{U}$ are distinguished by creating a disjoint copy of $\mathbf{U}$, denoted $\mathbf{U}'$, as we did when defining direct products.

Any collection f of $2^{|\mathbf{U}|}$ elements of type $< \mathbf{U}, \mathbf{U}' >$ can be treated as a morphism $\mathbf{U} \xrightarrow{f} \mathbf{U}'$, provided $\forall X \subseteq \mathbf{U}, \exists (X, Y) \in f$ (*i.e.* f is a total function) and the collection satisfies monotonicity, completeness and closure.[6] The ability to define a morphism/function in this element-wise fashion is a significant advantage of working with discrete spaces.

Recall that a category $\mathbf{C}$ is **cartesian closed** [7, 13] if

(1) there is a terminal object T in $\mathbf{C}$ such that for any object $A \in \mathbf{C}$ there is exactly one morphism $A \xrightarrow{f} T$;

(2) for any two objects $A, B \in \mathbf{C}$ there exists an object $A \times B \in \mathbf{C}$ with morphisms $A \times B \xrightarrow{\pi_A} A$, $A \times B \xrightarrow{\pi_B} B$ having the property that for any object C and morphisms $C \xrightarrow{f} A, C \xrightarrow{g} B$, there is a unique morphisms $C \xrightarrow{h} A \times B$ such that $h \cdot \pi_A = f$ and $h \cdot \pi_B = g$ (*i.e.* $\mathbf{C}$ has products); and

(3) for any two objects $A, B \in \mathbf{C}$ there exists an object B^A in $\mathbf{C}$ and a morphism $B^A \times A \xrightarrow{ev} B$ such that for each $C \times A \xrightarrow{f} B$ there exists a unique morphism

[5] If $|\mathbf{U}| = n$, there exist precisely 2^{n+1} such elements, half denoting the sets in the powerset, half denoting the identity morphism.

[6] Instead of monotonicity, closure, and completeness, one can have any set of characteristic properties provided only that they are preserved under composition.

$C \xrightarrow{g} B^A$ such that $(g \times id_a) \cdot ev = f$.
(The object B^A is called the *exponential object*).

$$\begin{tikzcd} \mathbf{B^A \times A} \arrow[r, "\mathbf{ev}"] & \mathbf{B} \\ \mathbf{C \times A} \arrow[u, dashed, "\mathbf{g \times id_A}"] \arrow[ur, "\mathbf{f}"] & \end{tikzcd}$$

Fig. 12. Exponential diagram.

Since objects in **ClosureSys** are appropriate collections of typed ordered pairs, to show **ClosureSys** is cartesian closed we must show that these objects/collections exist.

Theorem 2. *The category* **ClosureSys** *is cartesian closed.*

Proof. Since Ø is the terminal object and by Prop. 17 direct products exist, we need only show (3). Let $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ be closure systems over base sets $\mathbf{U}_A, \mathbf{U}_B, \mathbf{U}_C$ and suppose that $\mathcal{C}_C \times \mathcal{C}_A \xrightarrow{f} \mathcal{C}_B$. We must show there exists a unique morphism g into $B^A = [A \to B]$ (or a collection of category elements of type $< \mathbf{U}_A, \mathbf{U}_B >$) such that $(g \times id_A) \cdot ev = f$.
Consider an element $(X \uplus Y, Z)$ of type $< \mathbf{U}_C \uplus \mathbf{U}_A, \mathbf{U}_B >$ in the collection that is the morphism f. Let $(X \uplus Y).id_A = Y \in \mathcal{C}_A$, and let $X.g$ be the element (Y, Z) of type $< \mathbf{U}_A, \mathbf{U}_B > \in B^A$. With this construction, for any $X \uplus Y \in \mathcal{C} \times \mathcal{C}_A$, $(X \uplus Y).(G \times id_A).ev = (X.g \times Y).ev = Z = (X \uplus Y).f$. The only problem is to show that the collections of pairs $(Y, Z) \in B^A$ denoting $X.g$ really constitutes a monotone closed, complete function. This almost follows from Prop. 18, but it takes a bit more. We illustrate by demonstrating the monotonicity of $X.g$. Let $Y_1 \subseteq Y_2 \in \mathcal{C}_A$. Then, for any $X \in \mathcal{C}_C$, $X \uplus Y_1 \subseteq X \uplus Y_2$ in $\mathcal{C}_C \times \mathcal{C}_A$, and since f is monotone, $Z_1 \subseteq Z_2$. □

This construction is most easily visualized by its analog, the discrete Cartesian product of the integers $\mathbf{I} \times \mathbf{J}$. Let f be a function over $\mathbf{I} \times \mathbf{J}$, then for any $i \in \mathbf{I} < [i][\mathbf{J}].f$ denotes the "column" in the Cartesian array, which can be regarded as a function f_i of one integer variable. And f_i would be constructed analogously to Theorem 2, by collecting all function values $(i, k).f, k \in \mathbf{J}$. If f is, for example, non-decreasing using a city block metric then one can show that f_i is non-decreasing. This is important, because it is not the closure systems themselves that are "cartesian closed", but rather their morphisms.

The author wishes to acknowledge the valuable criticism of earlier drafts by G.Q. Zhang followed by stimulating conversations at Dagstuhl.

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