Jordan Surfaces in Discrete Antimatroid Topologies

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Abstract. In this paper we develop a discrete, T_0 topology in which (1) closed sets play a more prominent role than open sets, (2) atoms comprising the space have discrete dimension, which (3) is used to define boundary elements, and (4) configurations within the topology can have connectivity (or separation) of different degrees.
To justify this discrete, closure based topological approach we use it to establish an n-dimensional Jordan surface theorem of some interest. As surfaces in digital imagery are increasingly rendered by triangulated decompositions, this kind of discrete topology can replace the highly regular pixel approach as an abstract model of n-dimensional computational geometry.

1 Axiomatic Basis

Let $\mathbf{U}$ be a universe of arbitrary elements, or as we will call them, atoms. We let R denote a binary relation on $\mathbf{U}$. We denote the identity relation I on $\mathbf{U}$ by R^0. Relational composition is defined in the usual way, so $R^k = R \circ R^{k-1}$, and in particular, $R^1 \circ R^0 = R \circ I = R$. Notationally, we denote elements $(x, z) \in R^k$ by $x.R^k.z$.[3] Then, $x.R^k = \{z \mid x.R^k.z\}$ and $X.R^k = \{z \mid \exists x \in X, x.R^k.z\}$.

In addition to R, we assume an integer function $\delta : \mathbf{U} \to \mathbf{Z}$ that satisfies the following basic axiom

$$x.R.z \quad \text{implies} \quad \delta(x) > \delta(z). \tag{1}$$

An easy induction on k establishes that $x.R^k.z$ also implies $\delta(x) > \delta(z)$. Consequently,

Lemma 1. *If $x.R^m.z$ and $z.R^n.x$ then $n = m = 0$ and $x = z$.*

Proof. If $x.R^m.z$ then $\delta(x) > \delta(z)$, so if $z.R^n.x$ we have $\delta(z) > \delta(x)$, a contradiction unless $m = n = 0$ and $x = z$. □

Consequently, we can let

$$\delta(x) = min\{k \mid x.R^k \neq \emptyset, x.R^{k+1} = \emptyset\} \tag{2}$$

We should note that the implication of Lemma 1 coupled with the definition of δ in (2) together imply (1) and so these could be taken as the axiomatic basis instead.

Any relation R satisfying the functional constraint (1) is anti-symmetric. It is a pre-partial order; and its transitive closure $R^* = \bigcup_{k \geq 0} R^k$ is a partial order. Given such a relation R and function δ, we create a discrete topology τ by defining a closure operator φ. Since closure φ is just a relation on

[3] We employ a dot notation to clearly delineate the operator symbol from its argument(s).

$\mathbf{U}$, as is R, we use a similar kind of notation. A topology $\tau = (\mathbf{U}, R, \delta, \varphi)$ is said to be **locally finite** if for all atoms $x \in \mathbf{U}$, their closure $x.\varphi$ is finite. We will always assume that τ is locally finite.

Depending on one's choice of δ and φ, there are many varieties of discrete topology τ capable of describing the structure that R imposes on $\mathbf{U}$. For the closure operator φ on $\mathbf{U}$ we choose, in this paper, to use the ideal, or "downset", operator with respect to R, that is X is closed in $\mathbf{U}$ (with respect to φ), if $x \in X$ implies $z \in X$ for all $z \in x.R^k, k \geq 0$, or equivalently $X.\varphi = X.R^*$. Ideal operators are antimatroid, in that they satisfy the following anti-exchange axiom

$$\text{if} \quad x, y \notin Z.\varphi, \quad \text{then} \quad y \in (x \cup Z).\varphi \quad \text{implies} \quad x \notin (y \cup Z).\varphi. \tag{3}$$

(We often elide the braces { ... } around singleton sets in expressions such as $(\{x\} \cup Z)$). We can visualize any topological space $\tau^{(n)}$ as R on an n-partite space such as Figure 1.

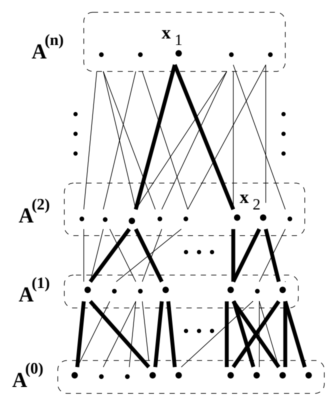

Fig. 1. An n-partite topological space, $\tau^{(n)}$.

We may assume $\delta : \mathbf{U} \to [0, n]$ because

Lemma 2. *If* $\tau = (\mathbf{U}, R, \delta, \varphi)$ *is locally finite and* φ *is the ideal operator on* R*, then* $\forall x \in \mathbf{U}, \exists k, x.R^k = \emptyset$.

Proof. Suppose not. Since $R^* = \bigcup_{k \geq 0} R^k$, $|x.R^*| = |x.\varphi| > k$ contradicting local finiteness. $\square$

By this definition, $\delta(\emptyset) = -1$, and those atoms x of $\mathbf{U}$ which are minimal with respect to R^* have $\delta(x) = 0$. If $\delta(x) = 1$, for every $y \in x.R$, $\delta(y) = 0$. More generally, if $\delta(x) = k$, then for every $y \in x.R$, $\delta(y) \leq k - 1$, and there exists at least one y, $\delta(y) = k - 1$. Clearly, $\delta : \mathbf{U} \to [0, n]$, so defined satisfies (1), and it is very reasonable in the context of the specific domain we will be examining later in this paper. As a notational convenience, we denote the collections of all atoms x, such that $\delta(x) = k$ by $A^{(k)}$. Thus the universe of all atoms $\mathbf{U} = \bigcup_{k=1,n} A^{(k)}$. An arbitrary collection Z of atoms we will call a **configuration** which we will denoted by $Z = [\, Z^{(0)}, Z^{(1)}, \ldots Z^{(n)}\,]$, where $Z^{(k)} = Z \cap A^{(k)}$. Sets and configurations we denote with uppercase letters; elements and atoms by lower case. A configuration Z is closed if $Z.\varphi = Z.R^* = Z$. In Figure 1, a closed configuration $Z = [\, \emptyset, \emptyset, \{x_2\}, \ldots, \{x_1\}\,].\varphi$ has been indicated by darker lines. We extend δ to configurations by letting $\delta(Z) = k$ where $k = max\{i \mid \delta(a) = i, a \in Z\}$. One can regard δ as a *dimension* concept.

2 Generators, Separation and Connectivity

A closure operator φ is a relation on $\mathbf{U}$ that is closed under intersection, that is $(X \cap Y).\varphi = X.\varphi \cap Y.\varphi$. Alternatively, it satisfies the standard closure axioms, *e.g.* it's monotone and idempotent, When the closure operator φ is defined by ideals in R, that is $\varphi = R^*$, these properties are evident. Moreover the anti-symmetry of R^* ensures it satisfies the anti-exchange property (3) and so φ is antimatroid. Equally important, this kind of ideal closure operator also has $(X \cup Y).\varphi = X.\varphi \cup Y.\varphi$, so it is also a "topological" closure operator.[4]

A topological space is T_0 if for any pair of points x and y, there exists at least one closed set containing one of them, but not the other [5]. Thus the reason for wanting the closure operator φ to be antimatroid is evident with the following theorem.

Theorem 1. *A discrete topology $\tau = (\mathbf{U}, R, \delta, \varphi)$ is T_0 if and only if its topological closure operator φ is antimatroid.*

Proof. Let φ be antimatroid. If we let $Z = \emptyset$ in (3), it immediately follows that τ is T_0.
Conversely, let $x, y \notin Z.\varphi$ and let $y \in (Z \cup x).\varphi$. We must show that $x \notin (Z \cup y).\varphi$. Since τ is T_0, there exists a closed set C containing precisely one of x or y, but not both. Suppose first that $x \in C$. Since $Z.\varphi \cup C$ is closed $(Z \cup x).\varphi \subseteq Z.\varphi \cup C$. But, now $y \notin Z.\varphi$ and $y \in (Z \cup x).\varphi$ imply that $y \in C$, contradicting choice of C. So, we must have $y \in C, \quad x \notin C$. Again, since $x.\varphi \cup C$ is closed, $(Z.\varphi \cup y).\varphi \subseteq Z.\varphi \cup C$. Then $x \notin Z.\varphi, \quad x \notin C$ imply $x \notin (Z \cup y).\varphi$ □

Theorem 2 will establish that the closed configurations of a discrete topology can be "shelled", one atom at a time.

Theorem 2. *Let $Z = [\,Z^{(0)}, Z^{(1)}, \ldots, Z^{(k)}, \ldots, Z^{(n)}\,]$ be a closed configuration of dimension k in $\tau^{(n)}$. For every atom, $z \in Z^{(k)}$, $Z - \{z\}$ is closed.*

Proof. Let $Z = [\,Z^{(0)}, Z^{(1)}, \ldots, Z^{(k)}, \ldots, Z^{(n)}\,]$ be any closed configuration of dimension k in $\mathcal{L}_\varphi$. Thus, $Z^{(k)} \neq \emptyset$, but for $\forall m > k, Z^{(m)} = \emptyset$. For any atom $z \in Z^{(k)}$, $Z - \{z\} = [\,Z^{(0)}, Z^{(1)}, \ldots, Z^{(k)} - z, \ldots, Z^{(n)}\,]$ and because $z \notin Z.R$, $(Z - z).R^* \subseteq Z - z$. Readily, $Z - z \subseteq (Z - z).R^*$, so $Z - z$ is closed. □

This is actually a well-known consequence of the antimatroid nature of φ in $\tau^{(n)}$. See [9, 11]. Alternately, Theorem 2 can be regarded as another proof that φ is antimatroid; one that is based solely on the definition of closure φ as an ideal R^*.

A set Y generates a closed set Z if $Y.\varphi = Z$. We say Y is a *generator* of Z, denoted $Z.\gamma$, if it is a minimal set that generates Z. The generator concept is fundamental in closure theory. For example, if closure is defined by a convex hull operator, then the generators of a convex polytope are its vertices. It is not hard to show that a closure operator is antimatroid if and only if every closed configuration has a unique minimal generator [11]. The set $X = \{x_1, x_2\}$ is the unique generator of the closed configuration of Figure 1. Many closure systems are not uniquely generated, therefore not antimatroid [4]. It is shown in [11] that

[4] The Kuratowski closure axioms [10] assume closure under union. This is not true for most closure operators.

Theorem 3. *Let Z be closed in an antimatroid space $\tau^{(n)}$. Y is a maximal closed subset of Z if and only if $Z - Y = \{x_i\}$, where $x_i \in Z.\gamma$.*

This has been called the "Fundamental Covering Theorem" since it completely defines the covering relationships in the lattice of closed subspaces of $\tau^{(n)}$.

It is evident from the definition of $\delta(Z)$ and of generators that $\delta(Z) = \delta(Z.\gamma)$. If $Z.\gamma \subseteq A^{(k)}$, we say Z is homogeneously generated, or just **homogeneous**. Readily $\delta(Z) = k$. The entire space $\tau^{(n)}$ must be closed; it is homogeneous if $\tau^{(n)}.\gamma \subseteq A^{(n)}$. Although the entire space as illustrated in Figure 1 is homogeneous, the closed set generated by $X = \{x_1, x_2\}$ is not.

Let $\tau^{(n)} = R$ over $A^{(0)}, A^{(1)}, \ldots, A^{(n)}$ be an antimatroid topology. The restriction of R to $A^{(0)}, \ldots, A^{(k)}$, $k < n$, denoted $\tau_k^{(n)}$, is called the k^{th} **subtopology** of $\tau^{(n)}$. If $\tau^{(n)}$ is the topology of Figure 1, then $\tau_2^{(n)}$ is just the lower tri-partite graph. Readily, if $\tau^{(n)}$ is homogeneous then $\tau_k^{(n)}$ is homogeneous.

A configuration Y is said to be **separable**, or **disconnected**, if there exist non-empty, disjoint, closed configurations Z_1, Z_2 such that $Y.\varphi = Z_1 \cup Z_2$.[5] A configuration Y is **connected** if it is not separable. A configuration X connects Y_1, Y_2 if $Y_1.\varphi \cap Y_2.\varphi = X \neq \emptyset$. Readily, only closed configurations can connect closed configurations.

This is just the classical sense of separation and connectivity cast in terms of closure. But, in discrete systems, it is often useful to consider connectivity of different "strengths". We say that X is k-**separable** if there exist closed configurations Z_1, Z_2 such that $X.\varphi = Z_1 \cup Z_2$, $\delta(Z_1 \cap Z_2) = k \geq -1$.[6] When $\delta(Z_1 \cap Z_2) = k < \delta(Z_i)$ we will say that $Z_1 \cap Z_2$ k-separates Z_1 and Z_2.

X is *k-connected* if it is not (k-1)-separable. X is **disconnected** if it is (-1)-separable, that is $Z_1 \cap Z_2 = \emptyset$. X is 0-connected if $Z_1 \cap Z_2 \subseteq A^{(0)}$.

Theorem 4. *X is k-connected if and only if $X.\varphi$ is k-connected.*

Proof. Let Z_1, Z_2 be closed configurations such that $Z_1 \cap Z_2$ (k-1)-separates X. $X = Z_1 \cup Z_2$. Since $Z_1 \cup Z_2$ is closed, $X.\varphi = Z_1 \cup Z_2$, so $Z_1 \cap Z_2$ also (k-1)-separates $X.\varphi$.
Proof of the converse is similar. □

Thus closure cannot increase connectivity. In particular, disconnected configurations cannot become connected by closure.

Lemma 3. *In $\tau^{(n)}$, if $\delta(X) = k \leq n$ then X is at most (k-1)-connected.*

Proof. Let $X.\varphi = Z_1 \cup Z_2$ where $Z_i, i = 1, 2$, is non-empty and closed. Readily, $\delta(Z_i) = \delta(X) = k$. Suppose $\delta(Z_1 \cap Z_2) = k$, that is there exists $x \in Z_1 \cap Z_2 \cap A^{(k)}$. Readily, $X.\varphi = (Z_1 - \{x\}) \cup Z_2$. By Theorem 2, $Z_1 - \{x\}$ is also closed. Use finite induction to remove all common atoms of dimension k until $\delta(Z_1 \cap Z_2) = k - 1$. □

If X is not (k-1)-separable, it cannot be (j-1)-separable, where $j < k$. So,

Lemma 4. *If X is k-connected, then X is j-connected for all $j \leq k$.*

From which it follows that

[5] A more customary definition would have $Y \subseteq Z_1 \cup Z_2$, with $Y \cap Z_k \neq \emptyset$ [5]. But, since $Z_1 \cup Z_2$ is closed and $Y.\varphi$ is the smallest closed set containing Y, this definition is preferable.

[6] Recall that in Section 1 we had defined $\delta(\emptyset) = -1$.

Lemma 5. *If X is i-connected in $\tau^{(n)}$, then X is i-connected in $\tau_k^{(n)}$, for all $0 < i \leq k$.*

Two atoms $x, z \in A^{(k)}$ are said to be **pathwise i-connected** if there exists a sequence $\rho_i = < y_0, \ldots, y_m >, m \geq 0$ such that $x = y_0, y_m = z$ and $y_j.\varphi \cap y_{j+1}.\varphi \cap A^{(i)} \neq \emptyset$. That is, y_j and y_{j+1} are at least i-connected. Pathwise connectivity can be regarded as a relation ρ_i on the atoms of the space with $(x, z) \in \rho_i$ if they are pathwise i-connected. Demonstrating that ρ_i is an equivalence relation is an easy exercise.

One would like to show that topological connectivity and pathwise connectivity are equivalent concepts, that is, a configuration X is topologically k-connected if and only if it is pathwise k-connected. Unfortunately, this is only partially true. To begin, it is easy to show that,

Theorem 5. *If a configuration X is pathwise k-connected it is topologically k-connected.*

Proof. Let X be pathwise k-connected and suppose there exists closed, non-empty Z_1, Z_2 such that $X = Z_1 \cup Z_2$ and $\delta(Z_1 \cap Z_2) \leq k - 1$. By Lemma 3, $\delta(X) = \delta(Z_1 \cup Z_2) > k$. Let $x \in Z_1, z \in Z_2$ where $\delta(x) = \delta(z) = k + 1$. Let $\rho_k =< y_0, \ldots, y_m >$ be a k connected chain of k+1 atoms between x and z which exists by hypothesis.
Since $y_0 \in Z_1, y_m \in Z_2$ there exists some pair of atoms $y_i \in Z_1, y_{i+1} \in Z_2$. But, $y_i.R \subseteq y_i.\varphi \subseteq Z_1$ and $y_{i+1}.R \subseteq y_{i+1}.\varphi \subseteq Z_2$. So $\delta(Z_1 \cap Z_2) = k > k - 1$, a contradiction. □

To see that the converse need not be true, consider the simple counter example of Figure 2. Readily the entire space $\tau^{(n)} = [\, A^{(0)}, A^{(1)}, A^{(2)} \,] = [\, \{x_1, x_2, x_3, x_4\}, \{y_1, y_2\}, \{z\} \,]$ is topologically

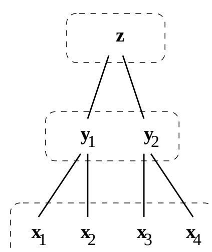

Fig. 2. A topologically connected configuration that is not pathwise 0-connected.

connected. But, the two 1-atoms y_1 and y_2 are not pathwise 0-connected.

A configuration X in $\tau^{(n)}$ is said to be **completely k-connected** if in each subtopology $\tau_i^{(n)}, i \leq k$, X is (i-1)-connected. If $X \in \tau^{(n)}$ is completely (n-1)-connected, we just say it is **completely connected**. Complete connectivity and pathwise connectivity are equivalent concepts because,

Theorem 6. *If a configuration Z is completely k-connected it is pathwise k-connected.*

Proof. This is most easily shown by the contrapositive. Suppose X is not pathwise k-connected. That is, there exists no sequence of (k+1) atoms between a pair of atoms $z_1, z_2 \in Z$. Consider the subtopology $\tau_{k+1}^{(n)}$. Let G_1 be the configuration of all (k+1)-atoms that can be reached by a k-path from z_1. Similarly, let G_2 be the configuration of all (k+1)-atoms reachable by a k-path from z_2.

Now, let $Z_1 = G_1.\varphi$ and $Z_2 = G_2.\varphi$. W.l.o.g we can assume $Z = Z_1 \cup Z_2$. (If not, we can form Z_3 in the same way from remaining (k+1)-atoms.) Readily, $\delta(Z_1 \cap Z_2) < k$ else there would be a k-path from z_1 to z_2. Z is not k-connected. □

We can use virtually the same proof to show that complete connectedness is inherited by subconfigurations.

Corollary 1. *If X is a k-connected subconfiguration of Z which is completely k-connected, then X is completely k-connected.*

Lemma 6. *Let $X \in \tau^{(n)}$. If X is completely k connected then $X.\gamma \subseteq A^{(k+1)} \cup \ldots A^{(n)}$.*

Proof. Suppose $x \in X.\gamma \cap A^{(i)}, i \leq k$. Since x is a generator, for all $j < i, x.R^j \cap X = \emptyset$, (*i.e.* x is maximal in the n-partite representation). But, then x can at most be (k-1)-connected to any other atom, contradiction assumption of k-connectivity □

As suggested by this lemma, one would like to be able to somehow equate homogeneity and complete connectivity. But this need not be true. Figure 3(a) is homogeneous, but not 1-connected. Figure 3(b) is completely 1-connected (by default), but not homogeneous. However, when $\tau^{(n)}$ is completely

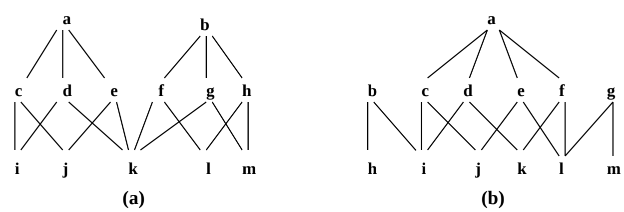

Fig. 3. Contrasting homogeneity and complete connection

(n-1)-connected we do have the corollary:

Corollary 2. *If X is completely connected in $\tau^{(n)}$, then X is homogeneously generated.*

In contrast to complete connectivity which is a global property, we have weak connectivity which is local. A configuration X with $\delta(X) = m$ is **weakly connected at** p if there exist atoms $x, z \in X$,

(a) $\delta(x) = \delta(z) = m$,
(b) $x.\varphi \cap z.\varphi = p$,
(c) $\delta(p) = k < m - 1$, and
(d) $X \cap p.R^{-1}.\varphi$ is not (k+1)-connected.

In Figure 3(a), $Z = \{a, b\}.\varphi$ is weakly connected at k. Figure 3(b) is not weakly connected at i because $\delta(b) = \delta(i) + 1 = 1$ violating condition (c) above. The point of this condition is to prevent the strongest possible connectivity from being called "weak".

3 Normality and Boundaries

A space is said to be *normal* if for all $k > 0, x \in A^{(k)}$ implies $x.R \subseteq A^{(k-1)}$ and $|\, x.R \,| \geq 2$. The topology of Figure 4 is non-normal at both atoms c and f. Heretofore all our examples have been

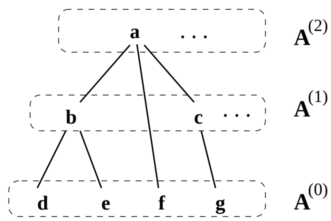

Fig. 4. A non-normal configuration.

normal, even though none of the proofs have required it. From now on we will assume that topologies are normal, even though it is still unnecessary for many of the results of this section.

By the **boundary** of a closed configuration Z, denoted $Z.\beta$, we mean that configuration of atoms, $Z.\beta = \{x \in Z.\varphi \mid x.(R^{-1})^* \not\subseteq Z\}$. We say $y \in Z.\beta$ is a **face** of Z if y is a generator of $Z.\beta$, that is $y \in Z.\beta.\gamma$.

Lemma 7. *$Z.\beta$ is closed.*

Proof. $Z.\beta.\varphi = \{x \mid \exists y \in Z.\beta, x \in y.R^*\}$. Since $y \in Z.\varphi$, $z \in z.R^*$ for some $z \in Z$, and by transitivity, $x \in z.R^*$. Further, since $y.(R^{-1})^* \not\subseteq Z$,and $y.(R^{-1})^* \subseteq x.(R^{-1})^*, x.(R^{-1})^* \not\subseteq Z$. So $x \in Z.\beta$. □

Lemma 8. *If y is a face of $Z.\beta$ then $y.R^{-1} \not\subseteq Z$*

Proof. $y \in Z.\varphi$ implies $y.R^{-1} \cap Z.\varphi \neq \emptyset$. Since $y.(R^{-1})^* \not\subseteq Z$, there exists $y' \in y.R^{-1}$ such that $y' \in Z.\beta$. $y \in y'.\varphi$ implies y is not a minimal generator. □

We often think of $Z.\beta$ as separating Z from its complement in $\tau^{(n)}$.

Lemma 9. *Let Z be any configuration in $\tau^{(n)}$, and let $W = \tau^{(n)} - Z$. Then $Z.\beta = W.\beta$.*

Proof. By definition $y \in Z.\beta$ if there exists $z \in Z$ and $w \in W$ such that $y \in z.R^* \cap w.R^*$. By symmetry, $y \in W.\beta$, and conversely. □

Lemma 10. *If $\tau^{(n)}$ is homogeneously generated and $\delta(Z) < n$, then $Z.\beta = Z.\varphi$.*

Proof. Readily $Z.\beta \subseteq Z.\varphi$. Conversely, $\forall y \in z.\varphi$, $y.(R^{-1})^* \cap A^{(n)} \neq \emptyset$ (since $\tau^{(n)}$ is homogeneous) so $y.(R^{-1})^* \not\subseteq Z$ implying $Z.\varphi \subseteq Z.\beta$. □

The boundaries of homogeneous configurations are homogeneous.

Lemma 11. *Let Z be a configuration in a completely connected space $\tau^{(n)}$. If $Z.\gamma \subseteq A^{(n)}$, then $Z.\beta.\gamma \subseteq A^{(n-1)}$.*

Proof. Let $Z.\gamma \subseteq A^{(n)}$. Suppose $b \in Z.\beta.\gamma \cap A^{(k)}, k < n-1$. $b \in Z.\beta \subseteq Z.\varphi$ implies there exists $a \in Z.\gamma \subseteq A^{(n)}$ such that $b \in a.\varphi$. $b \in Z.\beta$ also implies there exists $a' \notin Z.\varphi$. Because by Corollary 2, $\tau^{(n)}$ is homogeneous, we may assume w.l.o.g. that $a' \in A^{(n)}$. Now $a \in b.(R^{-1})^*$ and $a' \in b.(R^{-1})^*$. Since a and a' are k-connected (through b) and $\tau^{(n)}$ is completely connected, a and a' are (n-1)-connected, say through $c \in A^{(n-1)}$. By definition $c \in Z.\beta$, and by transitivity $b \in c.R^{-1}$, so b cannot be in $Z.\beta.\gamma$ □

The converse need not be true.

A generator x of a topology $\tau^{(n)}$ is said to be on the **border** of the space if there exists $y \in x.R$, such that $y.R^{-1} = \{x\}$. Let $\tau^{(n)}.B$ denote the collection of border generators. Note that $\tau^{(n)}.B$ may be empty. A configuration Z is said to be in **interior** position if $Z.\varphi \cap \tau^{(n)}.B.\varphi = \emptyset$.

Theorem 7. *Let Z be a completely connected configuration in interior position of $\tau^{(n)}$, $n \geq 2$, then $Z.\beta$ is completely (n-2)-connected.*

Proof. By induction on $|\, Z.\gamma \,|$. Readily, if $|\, Z.\gamma \,| = 1$, then $Z.\beta$ is (n-2)-connected because Z in interior position ensures that $Z.\beta \subseteq Z.\gamma.R^*$, and normality then ensures connectivity of $Z.\beta$.
To make the induction work, we must establish that when $|\, Z.\gamma \,| = n$ there exists a generator $x \in Z.\gamma$ which has a face in $Z.\beta$, whose removal will still leave Z (n-1)-connected.
The tricky induction step is when $|\, Z.\gamma \,| = 2$. Let $Z.\gamma = \{x_1, x_2\}$, where Z is completely connected. Because, Z is (n-1)-connected, Z is homogeneous (Corollary 2) with $x_1, x_2 \in A^{(n)}$, and $x_1.R \cap x_2.R = y \in A^{(n-1)}$; $y \notin Z.\beta$. But, since $\tau^{(n)}$ is normal, $x_2.R - \{y\} \neq \emptyset$, so x_2 has a face in $Z.\beta$. Readily, one can remove x_2 from Z so that $Z - \{x_2\}$ is still (n-1)-connected. Now, let $|\, Z.\gamma \,| = n$. Let $x \in Z.\gamma$ be any generator with a face in $Z.\beta$. If $Z - \{x\}$ is still (n-1)-connected, remove x. Otherwise, consider either of the two (n-2)-separated configurations Z_1 or Z_2, $Z_1 \cup Z_2 = Z$. $|\, Z_i.\gamma \,| < |\, Z.\gamma \,|$, so by induction there exists a generator $x \in Z_i.\gamma$ satisfying our requirements.
Remove x. $(Z - \{x\}).\beta$ is pathwise (n-2)-connected as is $\{x\}.\beta$ by induction. Since the faces common to $(Z - \{x\}).\beta$ and $\{x\}.\beta$ are each pathwise (n-2)-connected to the remaining faces, $(Z - \{x\} \cup \{x\}).\beta$ is pathwise (n-2)-connected. □

The converse need not be true. Even though a boundary is pathwise connected it may bound a weakly connected configuration. However we do know that:

Lemma 12. *If $Z.\beta$ is completely (n-2)-connected in $\tau^{(n)}$, then Z is at least (n-2)-connected.*

Proof. By Theorem 4, we may assume Z is closed. Since it is not (n-1)-connected, there exist two closed configurations Z_1 and Z_2 such that $Z = Z_1 \cup Z_2$, and $\delta(Z_1 \cap Z_2) \leq n - 2$. But, $Z.\beta \subseteq Z.\varphi = Z$, so $\delta(Z_1 \cap Z_2) \geq n - 2$. □

4 Geometric Spaces

In the rest of this paper we develop a specific discrete topology which is appropriate for digital images. It assumes an ideal closure and bounded dimension $\delta : \mathbf{U} \rightarrow [0, n]$. Its culmination will be another "Jordan Surface Theorem" which has attracted so much attention in the digital topology literature [3, 6–8]

Intuitively, a discrete n-dimensional, geometric space is formed by subdividing the space with (n-1)-dimensional constructs, whose intersections yield (n-2)-dimensional objects, *etc.* A 2-dimensional space is subdivided by lines which intersect in points, as in Figure 5(a). We will begin using geometric terms and call 0-atoms, "points"; 1-atoms, "lines". Instead of calling 2-atoms, regions, we prefer to use "tiles"; and instead of volumes, we will call 3-atoms "bricks". Computer applications, such as digital image processing, typically expect much more regular topologies such as Figure 5(b). In this field, "pixels" and "voxels" are a standard terminology. Atoms in $x.R$ and $x.R^{-1}$ are said to be **incident** to x. Thus, line 6 is incident to tile I and to the points b and h, but tiles I and II are not incident to each other. Terminology with a visual basis can help intuitive understanding; but

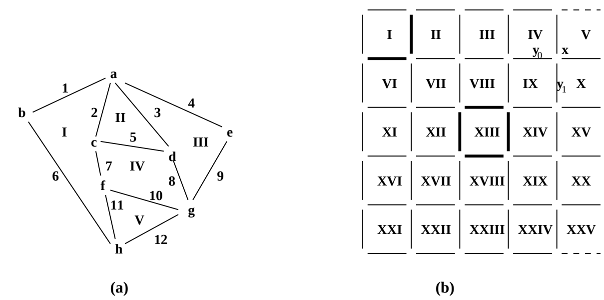

Fig. 5. Two geometric $\tau^{(2)}$ topologies.

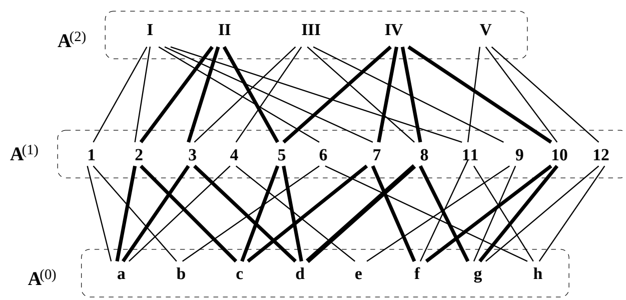

Fig. 6. The n-partite representation of Figure 5(a).

fundamentally, any discrete topology can still be represented as an n-partite graph such as Figure 6.

The notion of a boundary becomes more intuitive in geometric spaces such as Figures 5(a) and (b). Let $Z = \{II, IV\}.\varphi$ in Figure 5(a) corresponding to the darkened edges of Figure 6. Then $Z.\beta = [\,\{a,c,d,f,g\}, \{2,3,7,8,10\}, \emptyset]$. Observe that the line 5 is not in $Z.\beta$. Intuitively, an atom is in the boundary of Z only if it is incident to some atom not in Z. Readily, the generator of $Z.\beta$ is $Z.\beta.\gamma = \{2,3,7,8,10\}$. These are the faces of Z Only the generating tiles, two lines, and one point have been labelled in Figure 5(b). Here $\{XIII\}.\beta$ consists of the surrounding four bold lines; but $\{I\}.\beta$ consists of just the two bold lines. The remaining lines in $\{I\}.R$ are not incident to a tile "not in in Z". Tile I is a border tile of $\tau^{(n)}$. The atoms $\{1,4,6,9,12\} \subseteq A^{(1)}$ of Figure 6 are covered by singleton atoms. These singletons $\{I, III, V\} \in A^{(2)}$ constitute $\tau^{(n)}.B$.

A homogeneously generated n-dimensional topology $\tau^{(n)}$, will be called **geometric** if [7]

G1: $x \in y.R, y \in z.R$ implies there exists a unique $y' \neq y$ such that
$x \in y'.R$ and $y' \in z.R$.

G2: for all $k > 0, x \in A^{(k)}$ implies $|\,x.R\,| \geq k+1$.

G3: for all $k < n, x \in A^{(k)}$ implies $|\,x.R^{-1}\,| \geq n-k+1$, and

G4: $y \in A^{(n-1)}$ implies $|\,y.R^{-1}\,| \leq 2$.

Figure 7(a) illustrates the G1 property. Given the presence of line 3, with incident point c, that

[7] That G1 thru G4 are properties of "geometric" topologies can be easily verified. Whether they are sufficient to characterize these topologies is unknown.

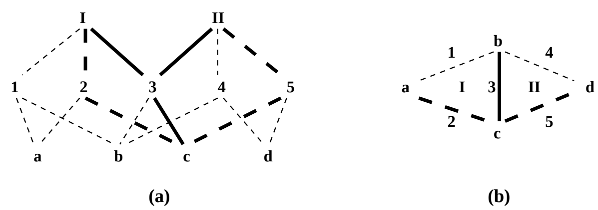

Fig. 7. Illustration of G1: (a) the n-partite representation of the geometry shown in (b).

is incident to tiles I and II then the existence of two more lines, which we have labelled 2 and 5, that are also incident to tiles I and II and point c is forced. The other lightly dashed lines in Figures 7(a) and (b), the geometric equivalent of Figure 7(a), denote possible configurations; but other boundaries of the tiles I and II are quite possible. Only the two lines 2 and 5 are forced.

Property G2 further strengthens the usual normal constraint of $|\,x.R\,| \geq 2$ if $\delta(x) > 0$. It says that each line must have at least 2 end points, each tile must have at least 3 bounding lines, and each brick must have at least 4 bounding tiles. This corresponds to simplicial decomposition of physical space as we normally view it into triangles and tetrahedrons. Condition G3 implies that in a 2-dimensional space, any point must be in the boundary (an endpoint) of at least 3 lines. Otherwise, the point is topologically redundant. Similarly, in a 3-dimensional space, each line segment must be incident to at least 3 tiles, because otherwise it too would be topologically redundant. Condition G4 which asserts that any (n-1)-atom can be the face of at most two atoms ensures that connected 1 and 2-dimensional topologies are strings and planar surfaces respectively. The closure structure of a geometric $\tau^{(1)}$ is the connected ordered topological space, or COTS, described in [6–8] and later illustrated in Figure 9. In 3, and higher dimensions, it asserts that a topological hyperplane separates exactly two regions.

Condition G3 says that (n-1)-atoms must separate at least two n-atoms, while condition G4 says they can separate no more than two n-atoms. Consequently, (n-1)-atoms must separate exactly 2 n-atoms, *except* possibly when the n-atom is at the *border*, $\tau^{(n)}.B$, of the entire space. Finite, discrete spaces often have borders where the expected properties of geometric spaces no longer hold. For example, in Figure 5(a), lines 1, 4, 6, 9 and 12 are incident to only one tile, and the points b and e are incident to only two lines. We must allow for these exceptions.

In Figure 5(a) and Figure 6, tiles I, III, V are border generators of $\tau^{(2)}.B$. There are no interior tiles in these two figures. In Figure 5(b), the 16 "outside" tiles constitute the border. All the remaining tiles satisfy the geometric constraints G1 through G4, but only tile $XIII$ is an interior tile. Generators in interior position are well removed from the border of $\tau^{(n)}$, if there is one. In the discussions that follow we will assume all configurations have only generating atoms in interior position.

We now focus on the properties of the boundaries $Z.\beta$ of configurations Z in interior position in geometric topologies.

Lemma 13. *Let Z be a homogeneous n-dimensional configuration in interior position in a geometric topology $\tau^{(n)}$. Let y be a face of $Z.\beta$ and let $x \in y.R$. There exists a unique $y' \neq y$ such that*

(a) $x \in y'.R$, and

(b) $y.R^{-1} \cap Z \subseteq y'.R^{-1} \cap Z$.

Proof. Since $y \in Z.\beta$, y separates some $z \in Z$ from some $w \in \tau^{(n)}-Z$. Because $y \in A^{(n-1)}$, $y.R^{-1} = \{w, z\}$, so $y.R^{-1} \cap Z = z$. Since $y \in z.R, x \in y.R$ the existence of a unique y' with $x \in y'.R$ follows from G1. $y' \in z.R$ assures (b) $\square$

Note that y' need not be an element of $Z.\beta$.

The following "continuation" theorem asserts that given a face $y_0 \in Z.\beta$, one can move in any "direction", *e.g.* across any face x of y_0, and find another face, possibly several, in $Z.\beta$. To see the import of this theorem, consider the two configurations of Figure 8. In $Z.\beta$ of Figure 8(a), line 2 in

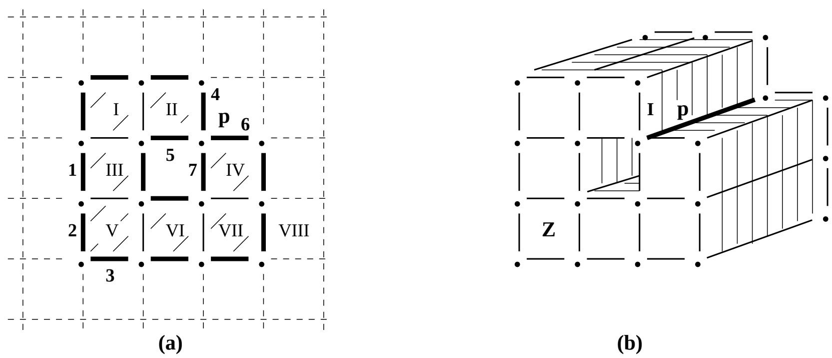

Fig. 8. Two configurations that are weakly connected at p.

$A^{(1)}$ is 0-connected to lines 1 and 3 in $Z.\beta$. Since $Z.\beta$ is defined with respect to the n-atoms of Z and $\tau^{(n)}-Z$, we cannot ignore the topology of these configurations as we investigate $Z.\beta$. In Figure 8(a) tile V is pathwise 1-connected to III and V (itself) of which 1 and 3 are faces. Similarly the corresponding tiles of the complement are pathwise connected. It is at places where Z is weakly (n-2) connected that possible complications can arise. Continuing face 4 of $Z.\beta$ through p could lead to faces 5, 6, or 7. Face 5 is pathwise connected *through* Z; and face 6 is pathwise *connected* through its complement. To make these ideas more formal we modify our path notation somewhat. Assuming the connectivity k is known, we now let $\rho_x(y_0, \ldots, y_n)$ denote a k-connected path such that $x \in y_i.\beta$ for $0 \leq i \leq n$. $\rho_x(y_0, \ldots, y_n)$ can be visualized as a path *around* x. In Figure 8(b) the tile labelled I in $Z.\beta$ is (n-2) path connected to three distinct tiles around p, two of which are hidden in the "tunnel".

Theorem 8. *Let Z be a homogeneous n-dimensional configuration in interior position in a geometric topology $\tau^{(n)}$, $n \geq 2$. Let y_0 be any face in $Z.\beta$ separating $z_0 \in Z$ from $w_0 \in W = \tau^{(n)}-Z$, and let $x \in y_0.R$. There exists a face y_n of $Z.\beta$ separating $z_n \in Z$ from $w_m \in W$, such that x separates y_0 from y_n and either*

(a) y_n is unique, in which case z_0, z_n are pathwise (n-1)-connected in Z and w_0, w_m are pathwise (n-1)-connected in W,

or else

(b) Z (and W) is weakly (n-2)-connected at x.

Proof. Let $W = \tau^{(n)}-Z$ and let y_0 separate $z_0 \in Z$ from $w_0 \in W$. Application of Lemma 13 using x, y_0, z_0 ensures the existence of a unique y_1 such that $x \in y_1.R, y_1 \in z_0.R$. Since $y_1 \in A^{(n-1)}, y_1$

separates z_0 from some $a \in A^{(n)}$.
If $a \in W$, y_1 is a face of $Z.\beta$. Let $y_n = y_1$.
If $a \in Z$, let $z_1 = a$ and iterate the application of Lemma 13 using x, y_i, z_i. By local finiteness, this construction must terminate with a face y_n separating $z_n \in Z$ from $w' \in W$ such that x separates y_0 from y_n. Observe that we have created an (n-1)-connected path $\rho_x(z_0, z_n)$ such that $x \in z_i.R^2, 0 \leq i \leq n$.
We now repeat this construction using x, y_0, w_0 to first obtain y'_1, and if necessary continue the construction to yield $y'_m \in Z.\beta$ separating $z' \in Z$ from $w_m \in W$. $\rho_x(w_0, w_m)$ is another (n-1)-connected path with $x \in w_j.R^2, 0 \leq j \leq m$.
If $y_m = y_n$ then $z' = z_n$ and y_n is a unique face.
If $y_m \neq y_n$, we let $z'_0 = z'$, $y'_0 = y_m$ and repeat the construction using x, y'_0, z'_0.
Eventually, we obtain a face $y'_k \in Z.\beta$ separating z'_k from $w'' \in W$ where $\rho(z'_0, z'_k)$ is (n-1)-connected and $x \in z'_i.R^2, 0 \leq i \leq k$.
Since weak connectivity is a local property, we need only observe that $\{z_0, z_1, \ldots z_n\} \subset Z$ is weakly connected to $\{z'_0, z'_1, \ldots z'_m\} \subset Z$ at x. □

The role of "interior position" in this theorem can be visualized using Figure 5(b). Suppose tile V has been deleted from the space and that Z consists of the central 9 tiles. Suppose y_0 is the face of IX separating it from IV. Rotating "counterclockwise" around x one gets y_1 separating IX from tile $X \in W$. But tiles IV and X in W are not pathwise connected because V is missing.

5 Jordan Surface Theorem

A traditional statement of the Jordan Curve Theorem is:

Theorem 9 (Jordan Curve Theorem). *Let C be a simple closed curve in $\mathbf{R}^2$. Then $\mathbf{R}^2 - C$ consists of exactly two components A and B. Moreover, $C = \overline{A} - A = \overline{B} - B$.[2]*

Is the essence of the "Jordan curve property" that of subdividing the space into precisely two components, with the purpose of the theorem to show that any simple closed curve in R^2 has this property; or rather is the property a constraint on the curve, which one then shows separates the space. Examples of both interpretations can be found in the literature. Our approach is to define a **discrete Jordan surface** in a geometric topology $\tau^{(n)}$, to be a configuration S that separates $\tau^{(n)}$ into precisely two pathwise (n-1)-connected components, W and Z such that $S = W.\beta = Z.\beta$, where one is in interior position.[8] Neither W nor Z need be simply connected. For example, a discrete torus floating in $\tau^{(3)}$ could be a Jordan surface. If $n \geq 3$ either W or Z can be locally weakly connected. The boundary $X.\beta$ of Figure 8(b) is a Jordan surface; the boundary in Figure 8(a) is not.

We briefly review the considerable history associated with configurations such as Figure 8(a); which has become known as the "Rosenfeld Paradox". The common assumption has been that a "curve" in a pixel space is a "thin" connected sequence of pixels, no more than one pixel wide. If Z, the foreground configuration of Figure 8(a), is regarded as a closed, 0-connected "curve" then it does not separate the background into two components because the complement too is 0-connected at p. If it is not a closed curve, *i.e.* not everywhere 1-connected, then it *does* separate the space.[9] A

[8] Requiring one component to be in interior position, by convention Z, eliminates analogs of hyperplanes from being considered as Jordan surfaces.

[9] We should note that Rosenfeld used the terms 8-connected and 4-connected instead of 0-connected and 1-connected [12]. This designated the number of "connected" pixels in a rectangular pixel space; it is standard in image processing.

common means of resolving the paradox is to use 0-connectivity for the foreground (or background) and 1-connectivity for its complement[12–14]. The paradox partially arises from the assumption that pixels are the only elements of the space. This is a perfectly natural assumption if one is analyzing digital images. Then Figure 8(a) may, or may not, be regarded as a "thin, continuous, closed curve" which may, or may not, be a Jordan curve separating the space.

Several authors have resolved this paradox by introducing spatial elements of lower dimension much as we have. One of us has begun with a connected ordered topological space (COTS), or Khalimsky topology shown in Figure 9. It has two kinds of element. It is T_0; it is antimatroid. Its

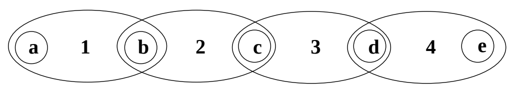

Fig. 9. A COTS, or Khalimsky topology.

direct product is equivalent to the geometric $\tau^{(2)}$ of Figure 5(b) ith the "pure" direct product of lines corresponding to tiles; the "pure" direct product of points would be a point; and the "mixed" direct products of a line with s point would be the same as our line [7, 8].

Although he only considers pixel elements, Gabor Herman [3] effectively introduces (n-1)-dimensional elements by considering the "boundary" of Z, denoted $\partial(Z, \mathbf{U}-Z)$, to be the collection of pixel pairs whose first element is in Z, with the second in $\mathbf{U}-Z$. When the topology is a pixel plane, these pixel pairs are equivalent to line segments; when it's a 3-dimensional array of voxels, it is a collection of voxel faces. It corresponds to our interpretation of a Jordan surface S in $\tau^{(n)}$ as being a boundary where $S.\gamma \subseteq A^{(n-1)}$.

Theorem 10. *Let Z be a homogeneous n-dimensional configuration in interior position in a geometric topology $\tau^{(n)}$. The boundary $Z.\beta$ is a Jordan surface if*
(a) $Z.\beta$ is completely (n-2) connected, and
(b) if Z is weakly (n-2) connected at $X \subset Z.\beta$, then there exists neither
a subset Z' of Z such that $X \subseteq Z'.\beta \subset Z.\beta$, nor
a subset W' of the complement, $W = \mathbf{U}-Z$, such that $X \subseteq W'.\beta \subset Z.\beta$.

Proof. (Necessity) If $Z.\beta$ is a Jordan surface, then both Z and W are completely (n-1)-connected, so (a) follows from Theorem 7 and Lemma 12 assures us that all points where $Z.\beta$ is (n-2)-connected, Z must be at least (n-2)-connected as well. The complete connectivity of Z and W ensures that neither Z' nor W' can exist at these points.
(Sufficiency) Because $Z.\beta = W.\beta$ is (n-2)-connected, both Z and W are at least (n-2)-connected by Lemma 12. Moreover, because Z is in interior position, our definition of "interior" ensures that all generators of W that are border atoms belong to a single (n-1)-connected component. Now suppose that Z (or W) is not (n-1)-connected. Say, $Z = Z_1 \cup Z_2$, where Z_1 and Z_2 are only (n-2)-connected. Then $Z_1, \beta \subset Z.\beta$. But, this is explicitly ruled out by condition (b). Thus $Z.\beta$ must be a Jordan surface. □

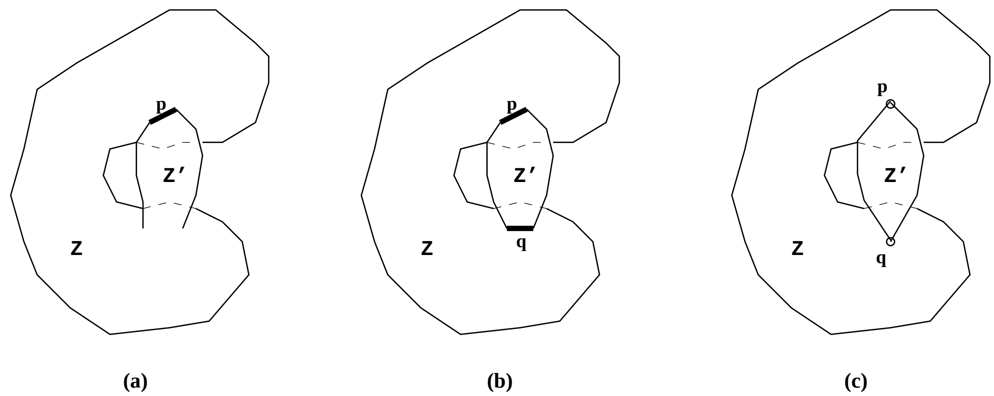

Fig. 10. Three configurations Z with different connectivities: (a) Z' is (n-1)-connected to Z, (b) Z' is (n-2)-connected to Z, (c) Z' is (n-3)-connected to Z.

Theorem 10 can be more easily visualized, where $n = 3$, by Figure 10. In Figure 10(a), $Z.\beta$ is 1-connected at p, but it is still a Jordan surface because no matter how the subconfiguration $Z' \subset Z$ is chosen, its boundary, $Z'.\beta \not\subset Z.\beta$. The portion at the base of Z' cannot be in the boundary of Z. In contrast, the boundary of Z in Figure 10(b) is not a Jordan surface because $Z'.\beta \subset Z.\beta$. Neither is the boundary of Z in Figure 10(c) a Jordan surface, but in this case it is because $Z.\beta$ is not (n-2)-connected.

Corollary 3. *Let Z be a homogeneous n-dimensional configuration in interior position in a geometric topology $\tau^{(n)}$. If Z is (n-1)-connected and nowhere weakly k-connected, where $k < n-1$, then $Z.\beta$ is a Jordan surface.*

Theorem 10 is of interest for two reasons. First, it completely characterizes all those surfaces which subdivide a geometric topology $\tau^{(n)}$ into precisely two completely (n-1)-connected components, one of which is in interior position. That component, which we have been denoting a Z, can be wildly contorted with many weak connectivities. But, we have shown that only at points of weak (n-2)-connectivity (where the argument of Theorem 8 shows the complement W must also be (n-2) connected) is further examination is required.

Second, it provides a theoretical basis for procedures to decide whether a specific surface of (n-1) atoms is a Jordan surface, thus reducing a potentially n-dimensional problem to one of (n-1) dimensions. Although provision of such an algorithm is beyond the scope of this paper, it is easy to envision one utilizing Theorem 8 which marks faces as examined and at each new face stacks all possible unmarked "adjacent" faces. For more sophisticated "face crawling" algorithms, the reader is refered to [3]. But, remember that those algorithms depend on a regular decomposition of the space. Our topologies need not be at all regular. They can easily arise from Voronoi decompositions [1] or polygonal mesh refinements [15].

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