A Category of Discrete Partially Ordered Sets

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Abstract

The most comon definition of the **Poset** category is incorrect. Here we present a valid categorical concept using morphisms that are "closed" and "complete".

1 Introduction

Except for simple sets, partially ordered sets, or posets, are probably the most commonly used discrete structures in mathematics. Yet, our knowledge of their transformational properties is quite incomplete. For example, in both [3, 10] we read that the collection of all posets, together with all order preserving morphisms f (that is, $x \leq y$ implies $f(x) \leq f(y)$) constitute a category **Poset**. But, the image of a poset under an order preserving map need not be a poset!

It is shown in [7] that if $P \xrightarrow{f} P'$ is an order preserving function on a poset P then its image P' is a poset if and only if for every point in P' its preimage is "convex" in P. Only order preserving functions with this additional property can be morphisms in **Poset**. However, we are getting ahead of ourselves. Let us start at the beginning.

2 Some Basic Concepts

2.1 Posets

Recall that a **partial order** is a relation $\leq$ on a set of points P such that $\leq$ is reflexive ($x \leq x$), weakly anti-symmetric ($x \leq y, y \leq x$ imply $x = y$), and transitive ($x \leq y, y \leq z$ imply $x \leq z$).[1] The set P, together with the partial order $\leq$, is commonly called a **poset**.

[1] There is an abundance of differing terminology. Some authors speak simply of "antisymmetry" [11]; some do not require reflexivity. Almost any transitive relation can be regarded as a "pre-order" lacking only the "anti-symmetric" property.

Alternatively, the poset concept can be cast is the terminology of directed graphs where we have a set of points, or nodes, P, a relation E on P, and a path relation ρ. A **path** $(x, z) \in \rho$ of length n is a sequence of points $< p_0, \ldots p_n >, n \geq 0$ such that $p_0 = x, p_n = z$, and $\forall i, 0 \leq i < n, (p_i, p_{i+1}) \in E$.[2] Readily, the path relation ρ is just the transitive closure of the edge relation E. A directed graph is said to be **acyclic** if $(x, z) \in \rho, (z, x) \in \rho$ imply $x = z$. So, acyclic graphs are completely isomorphic to discrete posets because reflexivity and transitivity have been assured by the definition of ρ. This correspondance becomes especially evident if one uses an infix path relation, so $x \, \rho \, z \equiv x \leq z$.

We prefer to work with the path relation ρ rather than $\leq$, because the former being constructed over the edge relation emphasises its discrete nature. It also distinguishes our development from that of continuous, partially ordered domains [1]. We draw posets (or acyclic graphs) from left to right just as one typically represents number lines. Figure 1 provides a typical example. We sometimes include arrowheads on the edges, as in this

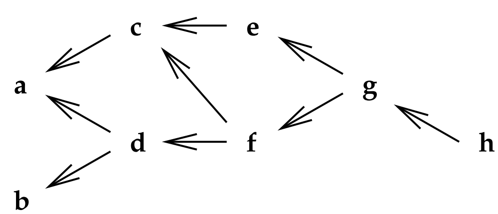

Figure 1: A representative poset.

illustration, to suggest the $<$ relation.

2.2 Homomorphisms

In [2, 6], a function $f : P \to P'$ is said to be a homomorphism of $G = (P, E)$ into $G' = (P', E')$ if $(x, y) \in E$ implies $(f(x), f(y)) \in E'$. Unfortunately, this common definition does not necessarily imply that G and G' have similar structure. For example, any totally ordered chain of elements is the homomorphic (order preserving) image of the anti-chain (in which $E = \emptyset$) on the same set of elements. We will say such an edge preserving function is simply **order preserving**. We will call f a **homomorphism** if in addition, $(x', y') \in E'$ implies $\exists x, y \in P$ such that $(x, y) \in E,\ f(x) = x', f(y) = y'$. Now, no ordered set can be the image of an anti-chain. Moreover, it ensures that any homomorphic monomorphism is an isomorphism. In particular, the example shown later as Figure 4 becomes impossible.

It is easily shown that

[2]We define paths in terms of a sequence of points in P rather than a sequence of edges in E simply so we can have a path of length zero from every point to itself.

Proposition 2.1 *If* $(P, E) \xrightarrow{f} (P', E')$ *is edge preserving then* $(x, z) \in \rho$ *implies* $(f(x), f(z)) \in \rho'$*. Or equivalently* $(P, \rho) \xrightarrow{f} (P', \rho')$ *is order preserving.*

While $(P, \rho) \xrightarrow{f} (P', \rho')$ is order preserving, we cannot conclude however that $(P, \rho) \xrightarrow{f} (P', \rho')$ is a homomorphism. Consider the function g of Figure 2 which is a homomorphism $(P, E) \xrightarrow{g} (P', E')$ but not a homomorphism $(P, \rho) \xrightarrow{g} (P', \rho')$, because because $(a', c') \in \rho'$, but there is no $(a, c) \in \rho$. We can strenghten the homomorphism concept a bit more by

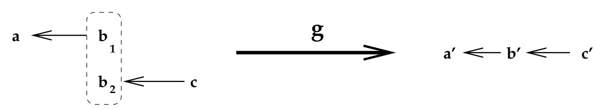

Figure 2: A homomorphism w.r.t. E that is not homomorphic w.r.t ρ.

requiring that if $(x', z') \in \rho'$ then $\exists x, z \in P$ such that $f(x) = x', f(y) = y'$ and $(x, z) \in \rho$. We call this a **path homomorphism**.

2.3 Why Posets with Order Preserving Morphisms are not a Category

It is not surprising that order preserving maps, in which $x\,R\,y$ implies $f(x)\,R\,f(y)$, are often introduced as early categorical examples; it is so easy to show that the composition is also order preserving. But, all that can be guaranteed is that the image will be a **pre-order**, that is reflexive and transitive. They need not be partial orders as is sometimes claimed [3, 4, 10]. Figure 3 illustrates a homomorphism f on an acyclic graph, or poset, whose image is not acyclic. Here, the only non-trivial pre-image subset of f is $\{b, h\}$. It is not difficult to see that f is really edge preserving, and that $(x, y) \in \rho$ implies $(f(x), f(y)) \in \rho'$. Even requiring f to be a path homomorphism is insufficient. Consider Figure 3. It is a

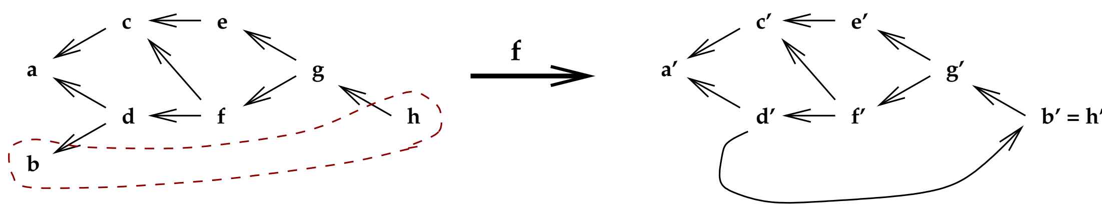

Figure 3: An order preserving homomorphism f.

path homomorphism because if there exists a path in P', say $(f', h') \in \rho'$, there exists a pre-image path in P, in this case $(f, b) \in \rho$.

Considering f as just a point function on P, we see it is an epimorphism, but not a monomorphism. However, it is easily shown that

Proposition 2.2 *The function* $(P, E) \xrightarrow{f} (P', e')$ *is order preserving if and only if* $E \subseteq E'$.

Consequently the monomorphism g on P shown as Figure 4 provides another, somewhat simpler, example of an order preserving function failing to preserve acyclicity.

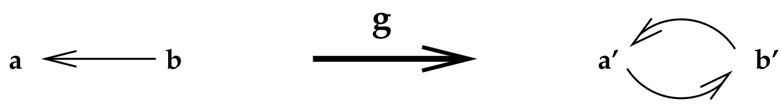

Figure 4: An order preserving monomorphism g whose image is not acyclic.

3 Closure and Closure Morphisms

Closure operators, φ, which are defined over sets of elements, play an important role in our development. Typically, we will be looking at the closure of subsets $X \subseteq P$ which we donote by $X.\varphi$ (read this as X closure). Recall, that closure operators must satisfy the three closure axioms (a) $X \subseteq X.\varphi$, (b) $X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$, and (c) $X.\varphi.\varphi = X.\varphi$. We call (P, φ) a **closure system** where φ can be any closure operator. A subset $Y \subset P$ is **closed** if $Y.\varphi = Y$. Given any closure operator φ, its closed subsets, partially ordered by inclusion, form a complete lower semi-modular lattice [5, 7].

While there exist many closure operators in the literature, we will focus on just three **path closures** φ_L, φ_R and φ_C, where for all $Y \subseteq P$,

$Y.\varphi_L = \{x \mid \exists y \in Y, (x, y) \in \rho\}$

$Y.\varphi_R = \{z \mid \exists y \in Y, (y, z) \in \rho\}$

$Y.\varphi_C = \{y \mid \exists y_1, y_2 \in Y, (y_1, y) \in \rho \text{ and } (y, y_2) \in \rho\}$

Observe that φ_L is sometimes called a "downset" operator, $\downarrow$, or ideal [1], and that φ_R is called an "upset" operator, $\uparrow$, or filter. The closed sets of φ_C are called "causal sets" by some theoretical physicists [12]. A set $Y \subseteq P$ is closed w.r.t. φ_C if and only if there exist subsets $X, Z \subseteq P$ closed w.r.t. φ_L and φ_R respectively, where $Y = X \cap Z$.

Path closures particularly nice because they are **anti-matroidal**; that is, if $x, y \notin Z.\varphi$ and $y \in (Z \cup \{x\}).\varphi$ then $x \notin (Z \cup \{y\}).\varphi$. In this case, the collection of *all* subsets partially ordered by

$$X \leq Y \text{ if and only if } Y \cap X.\varphi \subseteq X \subseteq Y.\varphi$$

forms a complete lattice [8].

Well-behaved functions over continuous domains are often characterized as "continuous" or "differentiable", where these properties are topologically defined in terms of open sets. In discrete closure systems, the open set concept seems forced. Instead, we can describe functions in terms of closure.

A morphism $(P, \varphi) \xrightarrow{f} (P', \varphi')$ is said to be **closed** if Y closed in P with respect to φ implies its image $Y.f$ is closed in P' with respect to φ'.[3] This property is clearly analogous to that of topologically open functions. The morphism $(P, \varphi) \xrightarrow{f} (P', \varphi')$ is said to be **complete** if $Y.f$ closed in P' with respect to φ' implies $Y.\varphi.f = Y.f$. That is, if Y maps onto a closed set Z', then its closure must map onto Z' as well. It turns out that completeness is analogous to continuity, although that is far from obvious.

Homomorphisms and order preserving maps were defined as point functions over the base sets with order preserving properties. These morphisms over closure systems are propertly "set-valued" maps defined over the subsets of P, or more precisely over the elements of the power set 2^P. By convention, we shall require all such morphisms to be **monotone**, that is $X \subseteq Y$ implies $X.f \subset Y.f$. Given monotonicity of f, it can be shown that [9]

Proposition 3.1 *A morphism $(P, \varphi) \xrightarrow{f} (P'\varphi')$ is closed if and only if $\forall X \subseteq P$, $X.f.\varphi' \subseteq X.\varphi.f$.*

Proposition 3.2 *A morphism $(P, \varphi) \xrightarrow{f} (P'\varphi')$ is complete if and only if $\forall X \subseteq P$, $X.\varphi.f \subseteq X.f.\varphi'$.*

So readily it follows that

Proposition 3.3 *A morphism $(P, \varphi) \xrightarrow{f} (P'\varphi')$ is closed and complete if and only if $\forall X \subseteq P$, $X.\varphi.f = X.f.\varphi'$.*

Closure and completeness were introduced in [9] as necessary and sufficient properties for the diagram of Figure 5 to commute.

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "\varphi"] & P' \arrow[d, "\varphi'"] \\
(P, \varphi) \arrow[r, "f"] & (P', \varphi')
\end{tikzcd}$$

Figure 5: f regarded as a closure system transformation

If $(P, \rho) \xrightarrow{f} (P', \rho')$ is an order preserving function we can "lift" it to subsets in the usual way[4]

[3]We prefer to use suffix functional notation when representing set valued morphisms such as these. It facilitates arrow chasing in categorical diagrams. We reserve more conventional prefix functional notation for point functions. The change of notation hopefully alerts the reader to the paradigm shift.

[4]This notation f^+ and f^- for set mappings that habe been lifted from functions is found in [13]. The maps f^- and f^{-1} are identical.

$Y.f^+ = \{y' \in P' \mid \exists y \in Y, y' = f(y)\}$
$Y'.f^- = \{y \in P \mid \exists y' \in Y', f(y) = y'\}$
In [2], the sets $\{y'\}.f^- = f^{-1}(y') = \{y \mid f(y) = y'\}$ are called the **fibers** of f, and the pre-image partition induced on P is called the **kernel** of f.

Proposition 3.4 *A function $(P, \leq) \xrightarrow{f} (P', \leq')$ is order preserving if and only if the lifted function $(P, \varphi) \xrightarrow{f^+} (P', \varphi')$ is complete with respect to a path closure.*[5]

Proof: Let φ be downset closure φ_L. Let f be a homomorphism and let $Y \subseteq P$. For all $y \in Y$, if $x \leq y$ then $x.f \leq y.f$ So $Y.\varphi_L.f^+ \subseteq Y.f^+.\varphi_L$ and by Prop. 3.2 f^+ is complete.
Conversely, assume f^+ is complete and that $x \leq y$. Then $x \in y.\varphi_L$ and $x.\varphi_L \subseteq y.\varphi_L$. Again by Prop. 3.2, $\{x\}.\varphi_L.f^+ \subseteq \{y\}.f^+.\varphi_L$ or $x.f \leq y.f$.
The proof when the closure is φ_C or φ_R is similar. □

So "complete" and "order preserving" are effectively synonyms. We use "order preserving" and "homomorphism" to describe point functions on P that also preserve the partial order. "Complete" describes a set-valued morphism $2^P \xrightarrow{f} 2^{P'}$, which may have been lifted from a point function. The following proposition suggests why completeness is analogous to continuity. Both preserve closed sets under inverse maps.

Proposition 3.5 *Let $(P, E) \xrightarrow{f} (P', E')$ be order preserving. If Y' is closed in (P', φ') w.r.t. any path closure φ', then $Y = Y.f^-$ is closed w.r.t φ.*

Proof: Since f^+ is order preserving, f^+ is complete. Since $Y.f^+$ is closed $Y.\varphi.f^+ = Y.f^+$ implying $Y.\varphi = Y$. □

Proposition 3.6 *Let $(P, \leq) \xrightarrow{g} (P', \leq)$. If $(P, \varphi) \xrightarrow{g^+} (P, \varphi')$ is closed with respect to a path closure then g is a path homomorphism.*

Proof: Assume that φ_L is the closure φ. Let $x \leq y$ or $x \in y.\varphi_L$, so $x.g^+ \in y.\varphi_L.g^+$. Since $y.\varphi_L$ is closed, $y.\varphi_L.g^+$ is closed, so $x.g^+ \in y.\varphi_L.g^+.\varphi_L$ or $x.g \leq y.g$. g is a homomorphism.
Now let $x' \leq y'$ in P', and let $\{y'\}.g^{-1} = Y$. $Y.g^+ = \{y'\}$ so $x' \in Y.g^+.\varphi_L \subseteq Y.\varphi_L.g^+$ implying there exists $y_k \in Y$ and $x_i \in Y.\varphi_L$ so $x_i \leq y_k$ where $x_i.g = x'$ and by definition of Y, $y_k = y'$. So g^+ is a path homomorphism.
Again, the proof is similar when the closure is φ_C or φ_R. □

Proposition 3.7 *Let (P, ρ) be finite, acyclic. Let $(P, \varphi) \xrightarrow{f^+} (P', \varphi')$ be closed with respect to any path closure. Then (P', φ') is acyclic.*

[5]Note that we use the traditional notation $f : \mathbf{U} \to \mathbf{U}'$ for point functions on $\mathbf{U}$, and $\mathbf{U} \xrightarrow{f^+} \mathbf{U}'$ for functions between the power sets.

Proof: Let $(X', y'), (y', x') \in \rho$, $x' \neq y'$ constitute a directed cycle. Let $X = \{x'\}.f^-$ and $Y = \{y'\}.f^-$ be distinct subsets in the kernel of f. Since f^+ is closed with respect to φ, but $X.f^+ = x'$ is not, X cannot be closed. In particular, there exists $x_0 \in X, y_0 \in Y$ such that $x_0 \in \{y_0\}.\varphi$. Since $\{y_0\}.f^+ = y'$ is not closed, $\exists x \in \{y_0\}.\varphi$, $x \in X$. Call it x_1. Since (P, ρ) is acyclic, $x_1 \neq x_0$.
Next we find $y_1 \in \{x_1\}.\varphi, y_1 \neq y_0$, and so forth. Since (P, ρ) is finite, this process must terminate, leading to a contradiction. $\square$

Although closed homomorphisms must be path homomorphisms, in general homomorphims that preserve acyclicity need not be closed w.r.t all path closures, as illustrated by f in Figure 6. The subset $\{d, b, a\}$ is closed with respect to φ_L, but its image $\{d', b', a'\}$ is

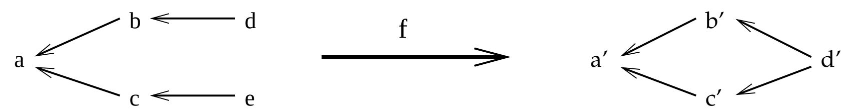

Figure 6: A homomorphism that is not closed w.r.t. φ_L.

not.

Even path homomorphisms that preserve acyclicity need not be closed as shown in Figure 7. g is a path homomorphism because $(x', y') \in \rho'$ implies $\exists(x, y) \in \rho$ in P. But, g is

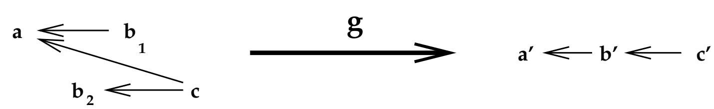

Figure 7: A path homomorphism that is not closed w.r.t. φ_L.

not closed. For instance, $\{b_2\}$ is closed with respect to φ_L in P but $b' = f(b_2)$ is not closed w.r.t. φ_L in P'.

By Prop. 3.7, we can create a **Poset** category by simply requiring its morphisms to be complete (*e.g.* order preserving) and closed. It is not hard to show that the composition of closed (complete) morphisms is closed (complete) [9]. But, as shown above, such a category would exclude many order preserving morphisms that also preserve acyclicity. We need to be more careful and concentrate on the φ_C closure operator.

If $Y \subset P$ is closed with respect to φ_C then every path $< p_0, p_1, \ldots, p_n >$ with $p_0, p_n \in Y$ lies completely within Y. That is, for all i, $p_i \in Y$. For this reason φ_C has been called a **convex closure** and its closed subsets called **convex** [7].

Proposition 3.8 *Let* $(P, \rho) \xrightarrow{f} (P', \rho')$ *be order preserving. Its image* (P', ρ') *is acyclic if and only if* $\forall y' \in P'$, $\{p'\}.f^-$ *is closed with respect to* φ_C.

Proof: That $(P, \rho'$ must be acyclic if f is closed follows from Prop. 3.7. Let (P', ρ') be acyclic. We claim that f^+ must be closed w.r.t. φ_C. It is sufficient to show that the fiber $\{y'\}.f^-$ of any $y' \in P'$ is convex in P. If not, there is a path $(y_0, y_1) \in \rho, y_0, y_1 \in Y$ and $x \notin Y$ such that $(y_0, x) \in \rho, (x, y_1) \in rho$. Now f order preserving implies $(y', x'), (x', y') \in \rho'$ contradicting acyclicity. $\Box$

4 The Category Poset

The category **Poset** consists of the collection Obj of all finite partially ordered sets, $(P, \leq)$ together with the collection Mor of all morphisms $(P, \leq) \xrightarrow{f} (P', \leq')$ that are complete and closed with respect to convex closure φ_C. Proposition 3.4 assures us that such morphisms are order preserving; Proposition 3.7 assures us that the codomains are all partial orders (acyclic) and that, moreover, every order preserving function g whose image is a poset (acyclic) is an element of Mor.[6]

Let $\mathcal{F} \subseteq Mor$ denote the family of all morphisms $(P, \leq) \xrightarrow{f} (P', \leq')$. Since the morphims of **Poset** are properly functions defined on the powerset 2^P, we can define a new morphism $f \cap g$ by $Y.(f \cap g) = Y.f \cap Y.g$, $\forall Y \subseteq P$.

Proposition 4.1 *Let $\mathcal{F}$ be the family of all complete, morphisms $(P, \leq) \xrightarrow{f} (P', \leq')$ then $f \cap g \in \mathcal{F}$.*

Proof: Readily, $f \cap g$ maps 2^P into $2^{P'}$. We need only show it is closed and complete.
If Y is closed, then $Y.f$ and $Y.g$ must be closed, so $Y.(f \cap g) = Y.f \cap Y.g$ is closed. Moreover, $f \cap g$ is complete because any edge $(x, y) \in E$ is preserved by both f and g (which are complete) and hence by $f \cap g$. $\Box$

Proposition 4.2 *Let $\mathcal{F}$ be a family of all closed, complete morphisms $(P, \leq) \xrightarrow{f} (P', \leq')$. $\mathcal{F}$ is a partially ordered set in Obj.*

Proof: Let $f, g \in \mathcal{F}$. Define $f \leq g$ if $f \cap g = f$. $\Box$

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[6] More properly it is the lifted version g^+ of g that is the element of Mor. We have tried to maintain the distinction; for example, the point function is order preserving, the lifted function is complete. But, it is easy to blur it.

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