Transformations of Discrete Closure Systems

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Abstract

Discrete systems such as sets, monoids, groups are familiar categories. The internal structre of the latter two is defined by an algebraic operator. In this paper we describe the internal structure of the base set **U** by a closure operator. We illustrate the role of such closure in convex geometries and partially ordered sets and thus suggest the wide applicability of closure systems.

Next, we look at functions f which map power sets 2^U into power sets $2^{U'}$, which we call transformations. In particular, we consider transformations which are monotone, continuous, or closed. These can be used to establish criteria for asserting that "the closure of a transformed image under f is equal to the transformed image of the closure".

Finally, we show that the categories **MCont** and **MClo** of closure systems with morphisms given by the monotone continuous transformations and monotone closed transformations respectively have concrete direct products. And the supercategory **Clo** of **MClo** whose morphisms are just the closed transformations is shown to be even cartesian closed.

1 Closure Systems

By a discrete system we mean a set of elements, points, or other phenomena which we will generically call our *universe*, denoted by **U**. Individual points of **U** will be denoted by lower case letters: $a, b, ..., p, q, ... \in \mathbf{U}$. If a finite set of points of **U** is given by listing them, we usually do not write commas between the letters denoting these individual points and,

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moreover, we often omit the set-denoting parentheses. So, for example, we write $\{abcd\}$ or even $abcd$ instead of $\{a, b, c, d\}$. By $2^{\mathbf{U}}$, we mean the powerset on $\mathbf{U}$, i.e., the collection of all subsets of $\mathbf{U}$. Elements of $2^{\mathbf{U}}$ we will denote by upper case letters: S, T, X, Y, Z. A **closure system** is a pair $(\mathbf{U}, \mathcal{C})$ where $\mathcal{C}$ is any collection of subsets of $\mathbf{U}$, including $\mathbf{U}$ itself, which is closed under arbitrary intersections. Subsets in $\mathcal{C}$ are said to be **closed**. Since the universe $\mathbf{U}$ is contained in $\mathcal{C}$, $\mathcal{C}$ itself will be often be regarded as the closure system. In this case, $\mathcal{C}$ is said to be a closure system on $\mathbf{U}$ and $\mathbf{U}$ is said to be the base universe of $\mathcal{C}$, in symbols $|\mathcal{C}| = \mathbf{U}$. For example, if $\mathbf{U} = \{a, b, c, d, e\}$ then the collection of closed sets

$$\mathcal{C}_{\mathbf{1}} = \{\emptyset, \{a\}, \{b\}, \{ab\}, \{bd\}, \{abc\}, \{abd\}, \{abce\}, \{abcde\}, \{abcdef\}\} \quad (1)$$

is a closure system.

A closure system can equivalently be defined as a pair $(\mathbf{U}, \varphi)$, where φ is a **closure operator** on $\mathbf{U}$, i.e., a self-mapping of $2^{\mathbf{U}}$, satisfying the following three axioms: For all $Y, Z \subseteq \mathbf{U}$,

C1: $Y \subseteq Y.\varphi$,

C2: $Y \subseteq Z$ implies $Y.\varphi \subseteq Z.\varphi$, and

C3: $Y.\varphi.\varphi = Y.\varphi$.

Here we are using a suffix operator notation, as we will throughout this paper. Read $Y.\varphi$ as "Y closure". A set Y is **closed** if $Y = Y.\varphi$. By C1, $\mathbf{U}$ itself must be closed. It is well known that these two definitions of closure are equivalent: The closed sets given by a closure operator on $\mathbf{U}$ form a closure system on $\mathbf{U}$ and, conversely, given a closure system $(\mathbf{U}, \mathcal{C})$, the closure operator on $\mathbf{U}$ is obtained by assigning to every subset $A \subseteq \mathbf{U}$ the intersection of all members of $\mathcal{C}$ that contain A as a subset.

Let $Y \subseteq \mathbf{U}$ be a closed set. A subset $X \subseteq Y$ is said to **generate** (or to be a **generating set** of) Y if $X.\varphi = Y$. By C3, every closed set Y generates itself. But, that tends to be uninteresting. We say a generating set X of Y is **non-trivial** if $X \subset Y$. In fact, we are really only interested in *minimal* non-trivial generating sets. More specifically, if X is a minimal generating set, it is called a **generator** of Y. A generator of Y under consideration will usually be denoted by $Y.\gamma$.[1] Of course, if Y is infinite, then it need not have any generator. When there is just a single generator for any closed subset of $\mathbf{U}$, we say that $(\mathbf{U}, \varphi)$ is **uniquely generated**. A closure system $(\mathbf{U}, \varphi)$ is said to be **finitely generated** if for all closed subsets of $\mathbf{U}$, their generators exist and are finite.

A closure operator/system can satisfy other axioms depending on the mathematical discipline. A **topological closure** is grounded and closed under finite union, i.e., satisfying

C0: $\emptyset.\varphi = \emptyset$,

C4: $(Y \cup Z).\varphi = Y.\varphi \cup Z.\varphi$.

The closure operator of linear systems, often called the **spanning operator**, satisfies the

[1] When a closed set Y has more than one generator, the collection of all of them we denote by $Y.\Gamma = \{Y.\gamma_1, \ldots, Y.\gamma_n\}$ [11].

Steinitz-MacLane **exchange axiom**

C5: if $p, q \notin Y.\varphi$, and $q \in (Y \cup \{p\}).\varphi$ then $p \in (Y \cup \{q\}).\varphi$.

Such closure systems are called **matroids**. Still other closure operators may satisfy an **anti-exchange axiom**

C6: if $p, q \notin Y.\varphi$, $p \neq q$, and $q \in (Y \cup \{p\}).\varphi$ then $p \notin (Y \cup \{q\}).\varphi$.

These closure operators, which include the geometric convex hull operator, are said to be **antimatroid closure operators** and the corresponding closure systems are called **antimatroids**. An important antimatroid property is that:

Proposition 1.1 *Let $(\mathbf{U}, \varphi)$ be an antimatroid, let $Z \subseteq \mathbf{U}$ be a closed subset and let X, Y be generating sets of Z. If each of the sets X and Y contains a finite subset generating Z, then $X \cap Y$ is a generating set of Z too.*

Proof: We have $X.\varphi = Y.\varphi = Z$. Let $A \subseteq X$ and $B \subseteq Y$ be finite generating sets of Z and suppose that $X \cap Y$ does not generate Z, i.e., that $(X \cap Y).\varphi \subset Z$. Put $C_0 = X \cap Y$. Then $A \not\subseteq C_0.\varphi$. Let $x_1 \in A - C_0.\varphi$ be an arbitrary point and put $C_1 = C_0 \cup \{x_1\}$. If $C_1.\varphi \subset Z$, then $A \not\subseteq C_1.\varphi$. Let $x_2 \in A - C_1.\varphi$ be an arbitrary point and put $C_2 = C_1 \cup \{x_2\}$. Repeating this construction, after a finite number k of steps we get a set $C_k = C_{k-1} \cup \{x_k\}$ such that $C_{k-1}.\varphi \subset Z$, $x_k \notin C_{k-1}.\varphi$ and $C_k.\varphi = Z$. Put $C_{k-1} = D_0$. Then $B \not\subseteq D_0.\varphi$. Let $y_1 \in B - D_0.\varphi$ be an arbitrary point and put $D_1 = D_0 \cup \{y_1\}$. If $D_1.\varphi \subset Z$, then $B \not\subseteq D_1.\varphi$. Let $y_2 \in B - D_1.\varphi$ be an arbitrary point and put $D_2 = D_1 \cup \{y_2\}$. Repeating this construction, after a finite number l of steps we get a set $D_l = D_{l-1} \cup \{y_l\}$ such that $D_{l-1}.\varphi \subset Z$, $y_l \notin D_{l-1}.\varphi$ and $D_l.\varphi = Z$. Thus, since $(D_{l-1} \cup \{y_l\}).\varphi = Z$, we have $x_k \in (D_{l-1} \cup \{y_l\}).\varphi$. But $(D_{l-1} \cup \{x_k\}).\varphi = Z$ holds too (because $(C_{k-1} \cup \{x_k\}).\varphi = Z$ and $C_{k-1} \subseteq D_{l-1}$), so we have $y_L \in (D_{l-1} \cup \{x_k\}).\varphi$. Suppose that $x_k \in D_{l-1}.\varphi$. Then $\{x_k\} \cup D_{l-1} \subseteq D_{l-1}.\varphi$, hence $(\{x_k\} \cup D_{l-1}).\varphi \subseteq D_{l-1}.\varphi$. It follows that $D_{l-1}.\varphi = Z$, which is a contradiction. Hence, $x_k \notin D_{l-1}.\varphi$. This contradicts the assumption that $(\mathbf{U}, \varphi)$ is an antimatroid. Therefore, $(X \cap Y).\varphi = Z$ and the proof is complete □

To see why the existence of finite generating subsets of X and Y is required in the previous statement, consider $\mathbf{U} = \mathbf{Z}$ (the set of integers) and let φ be the downset closure on the (naturally) ordered set $\mathbf{Z}$ (see section 1.3). Let X be the set of even integers and Y be the set of odd integers. Then both X and Y generate $\mathbf{U}$ but $X \cap Y$ does not because $(X \cap Y).\varphi = \emptyset.\varphi = \emptyset$.

A closure system is said to be **finitely generated** if every closed set has a finite generator.

Proposition 1.2 *Let $(\mathbf{U}, \varphi)$ be a finitely generated closure system. Then $(\mathbf{U}, \varphi)$ is an antimatroid if and only if $(\mathbf{U}, \varphi)$ is uniquely generated.*

Proof: If $(\mathbf{U}, \varphi)$ is not an antimatroid, then there exists some closed set $Y.\varphi$ with $p, q \notin Y.\varphi, p \neq q$, such that $p \in (Y \cup q).\varphi$ and $q \in (p \cup Y).\varphi$. Then p and q are members of distinct generators of

$(Y \cup p).\varphi = (Y \cup q).\varphi$. The converse is a corollary of Prop. 1.1. □

It follows from the previous statement that a finite closure system $(\mathbf{U}, \varphi)$ is an antimatroid if and only if it is uniquely generated. This is a useful result because most closure systems occurring in applications are finite.

If $(\mathbf{U}, \varphi)$ is a finite antimatroid, then the collection of all sets X_i with the same closure $Y = X_i.\varphi$ constitute a Boolean lattice with $Y.\gamma \subseteq X_i \subseteq Y$. In particular, if $X_i, \ldots X_k$ generate Y then $X_i \cap \ldots \cap X_k$ also generates Y by Proposition 1.1.

Let $(\mathbf{U}, \varphi)$ be a closure system, say $\mathcal{C}_1$ of (1). Containment, $\subseteq$, forms a natural partial ordering on the closed subsets $\mathcal{C}_1$. It is well known that the closed sets in $(\mathbf{U}, \varphi)$ so ordered form a lower semi-modular lattice.[2] Figure 1 illustrates the lattice representation $\mathcal{L}$ of the closure system $\mathcal{C}_1$. The sublattice of closed sets in this figure is denoted by set labels that

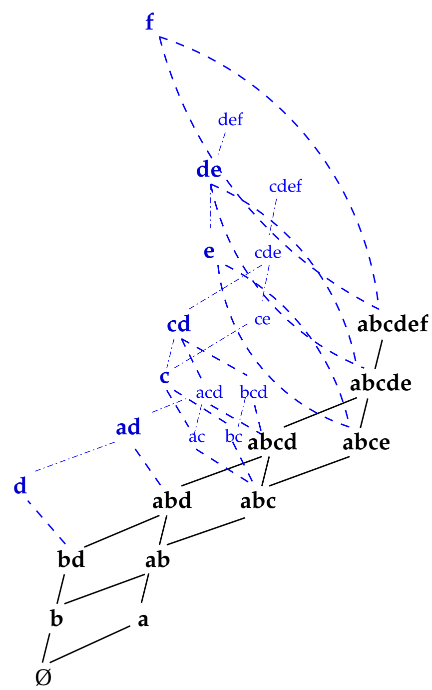


Figure 1: The lattice of closed sets of $\mathcal{C}_1$ with generating sets indicated

are joined by solid lines that are generally inclined from the upper right to the lower left. In Figure 1, the top of the sublattice has been pushed to the right to more easily draw the Boolean generator sets in an upper left to lower right orientation.

[2]This lower semimodularity of closed subsets partially ordered by inclusion has been repeatedly discovered by many authors. See Monjardet [16] for an interesting summary.

The generators, d, ad, c, cd, e, de and f, are connected to the corresponding closed sets that they generate by dashed lines generally inclined from the upper left to the lower right. Observe that each of the lattice intervals $[Y.\varphi, Y.\gamma]$ is a Boolean sublattice. In the case of the 32 subsets comprising the Boolean sublattice $[abcdef, f]$ and the 8 subsets comprising $[abcde, de]$ and $[abce, e]$, we only indicate a few of their their constituent elements and a dashed outline.

The regularity of structure suggested by this figure really exists, *c.f.* [18].

1.1 Relative Closure

In our development of closure space functions we will also use the following concept, which has the feeling and flavor of relative topologies. Let $(\mathbf{U}, \varphi)$ be any closure system and let $W \subseteq \mathbf{U}$. By the **relative closure** φ **with respect to** W, denoted by $\varphi|_W$, we mean

$$Y.\varphi|_W = Y.\varphi \cap W, \quad \forall\, Y \subseteq W.$$

It is not hard to see that if φ is an antimatroid, then $\varphi|_W$ is as well.

As shown below, the restriction of a closed set will always be closed. The ability to infer that Y is closed wrt. φ when its restriction is closed wrt. $\varphi|_W$ is of more interest. The following proposition from [18] gives two sufficient conditions. Neither is necessary.

Proposition 1.3 *In any closure system* $(\mathbf{U}, \varphi)$

(a) X *closed wrt.* φ *implies* $X \cap W$ *is closed wrt.* $\varphi|_W$.

(b) If W *is closed wrt.* φ *and* $X \subseteq W$, *then* X *closed wrt.* $\varphi|_W$ *implies* X *is closed wrt.* φ.

(c) If $X \subseteq W$ *is closed wrt.* $\varphi|_W$ *and* $X.\varphi \subseteq W$, *then* X *is closed wrt.* φ.

Closure systems are common in discrete applications, one encounters them in many guises. Three important types of closure operators are 1) convex hull operators, 2) poset closures, and 3) Galois closure of relations.

1.2 Convex Geometries

By a *convex geometry* we shall mean a collection of discrete points in an n-dimensional Euclidean space. For the illustrative purposes of this paper, we always assume a 2-space. The closure operator is the convex hull operator. A much more complete treatment can be found in [7, 8, 12]. There the closed sets are called *alignments*.

A convex hull closure operator provides the quintessential example of antimatroid closure. Consider Figure 2. The point q is in the convex hull of $Y \cup \{p\}$, consequently p cannot possibly be in the convex hull of $Y \cup \{q\}$.

A closed convex hull is generated by elements of its boundary. The minimal generating set is precisely the vertices of the boundary. Since, in a discrete geometry, these must be unique, convex hull closure is antimatroid. The efficiency of the Simplex method for

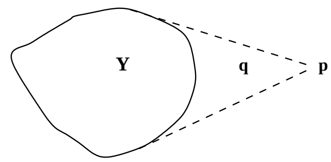


Figure 2: Illustration of antimatroid closure

solving linear inequalities [4] is based on this principle. In Figure 3 there is a small 6 point geometry. Every singleton point or doubleton pair is a trivial closed set. This is reflected in the "busy" structure of the closure lattice to the right. The non-trivial generating sets are $\{abcd, abcf, abdf,\ acde, bcde, abc,\ abd, abf, acd,\ bcd, cde\}$. In [7, 8, 12] they are called

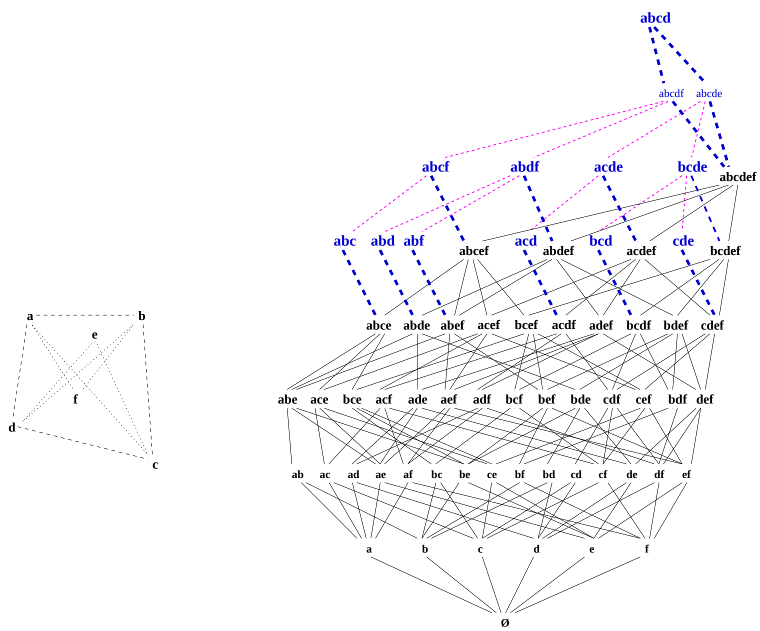


Figure 3: A small 6 point convex geometry

extreme points. The set of points *abcd* generate the entire space $(\mathbf{U}, \varphi)$.

1.3 Posets as Closure Systems

Suppose $\mathbf{U}$ is a poset $(P, \leq)$ such as Figure 4, where $a < c < e$, *etc.* Or, equivalently, we may think of Figure 4 as an acyclic graph $G = (P, E)$. Two familiar closure operators on P are

$Y.\varphi_{\downarrow} = \{\, x \mid \exists y \in Y, x \leq y \,\}$ and

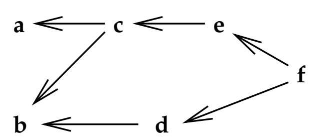


Figure 4: A 6 point poset

$Y.\varphi_{\uparrow} = \{\, z \mid \exists y \in Y, y \leq z \,\}$.

"Downset closure", $\varphi_{\downarrow}$, on Figure 4 yields precisely the closure system $\mathcal{C}_1$ introduced in Section 1 and illustrated in Figure 1.[3] $Y.\varphi_{\downarrow}$ is sometimes called an ideal, and $Y.\varphi_{\uparrow}$ a filter. In this paper we will emphasize the **downset** closure, $\varphi_{\downarrow}$, on posets; although others are possible.

A potentially more interesting closure operator on posets is the **convex closure** operator, denoted by φ_C and defined by $Y.\varphi_C = \{\, x \mid \exists y_1, y_2 \in Y, y_1 \leq x \leq y_2 \}$. Some authors call these closed sets "causal sets" [2]. The lattice of closed sets generated by φ_C operating on the 5 point poset of Figure 4 is shown in Figure 5. Observe how many more closed subsets

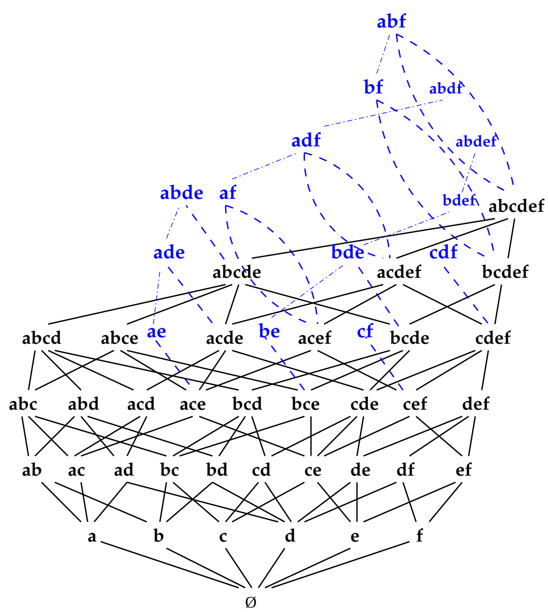


Figure 5: The lattice of closed sets formed by convex closure, φ_C on the poset of Figure 4 together with their generators.

of $P = \{a, b, c, d, e\}$ are created by φ_C than by $\varphi_{\downarrow}$. It is easy to show that $\varphi_C = \varphi_{\downarrow} \cap \varphi_{\uparrow}$.

[3] Since we draw partial orders from left to right, perhaps this should be called a "left set" ordering!

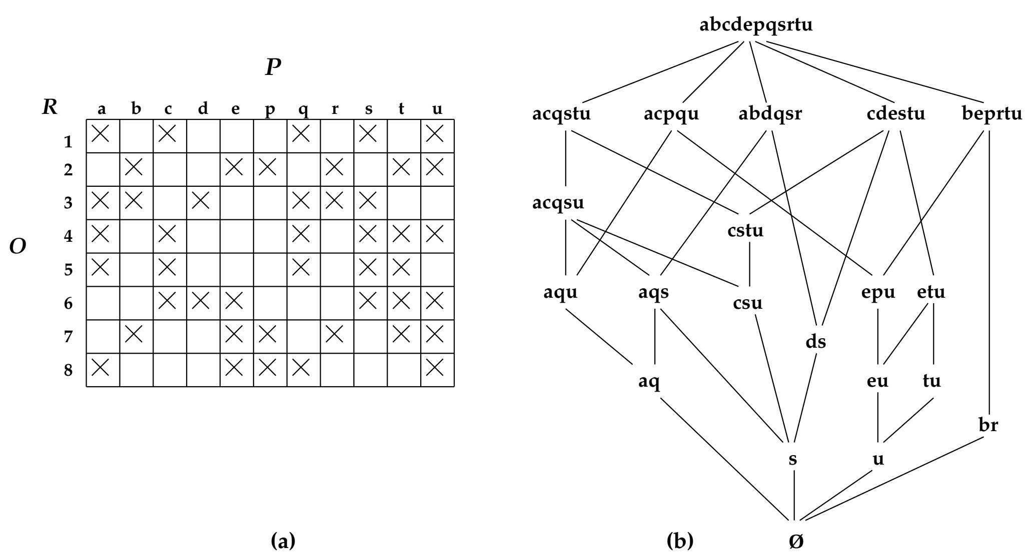
We call $\varphi_\downarrow, \varphi_\uparrow$, and φ_C collectively **path closures**.

All of the path closures $\varphi_\downarrow, \varphi_\uparrow$ and φ_C are antimatroid closures, and $\varphi_\downarrow$ and $\varphi_\uparrow$ are, in addition, topological closures.

1.4 Galois Closures

Suppose $\mathcal{O}$ denotes a set of "objects" and $\mathcal{P}$ denotes a set of "properties" that may, or may not, be attributes of the objects in $\mathcal{O}$. A Galois connection between $\mathcal{O}$ and $\mathcal{P}$ induces closure systems on $\mathcal{O}$ and $\mathcal{P}$. Each pair (O_i, P_i) of closed subsets $O_i \subseteq \mathcal{O}$ and $P_i \subseteq \mathcal{P}$ is called a concept in [9]. They denote the maximal set O_i of objects o_l that are common to every property p_m of P_i and the maximal set P_i of properties p_k that are common to every object o_n of O_i. Because of this, we can interpret the closure operator on a set of generators as a logical implication. That is, $(\forall o_n \in O_i)[P_i.\gamma \rightarrow P_i]$. The universal quantification asserts the implication is true for all observations so far, in which the generating events $P_i.\gamma$ are present.

For example, let $\mathcal{P} = \{a, b, c, d,\ e, p, q, r,\ s, t, u\}$ be properties that can be attributes of the 8 kinds of objects as shown in Figure 6(a). In "Formal Concept Analysis" [9] it is customary to display the lattice of closed sets in $\mathcal{P}$ as is shown in Figure 6(b).



R	a	b	c	d	e	p	q	r	s	t	u
1	×		×				×		×		×
2		×			×	×		×		×	×
3	×	×		×			×	×	×		
4	×		×				×		×	×	×
5	×		×				×		×	×	
6			×	×	×				×	×	×
7		×			×	×		×		×	×
8	×				×	×	×				×

Figure 6: Concept lattice resulting from the Galois closure on R.

The generators of a Galois closure are seldom unique, so they are seldom antimatroids. However, the collection of generators can be easily determined [11]. Let Z be a closed concept that covers the concepts Y_i in $\mathcal{L}_R$. Recall that Z **covers** Y_i if $Y_i \subset Z$ and $Y_i \subseteq$

$X \subset Z$ implies $Y_i = X$. If $X \subseteq Z$ is a minimal subset such that $X \cap (Z - Y_i) \neq \emptyset$ for all i, then X is a generator of Z.

The example of Figure 6 illustrates the Galois closure of a binary relation R, we have shown its application with n-ry relations $R^{(n)}$ as well, [23]

Galois closure can be a useful tool in data mining [10, 22]. We have used it to mine certain kinds of biological data [20] and software trace data [19]. For other applications, see [5, 9].

1.5 Partially Ordering a Power Set with respect to Closure

In the preceding sections, we partially ordered only the closed sets or generators by set inclusion. Now we seek to partially order *all* the subsets of $\mathbf{U}$. Simple set inclusion no longer seems quite so natural. In particular, set inclusion does not involve the closure operator. To partially order $2^{\mathbf{U}}$ with respect to the closure φ, we put

$$X \sqsubseteq Z \quad \text{if and only if} \quad Z \cap X.\varphi \subseteq X \subseteq Z.\varphi \tag{2}$$

for all $X, Z \subseteq \mathbf{U}$, It is not hard to show that $\sqsubseteq$ really is an ordering; only transitivity takes a bit of thought [18].

First, we observe that if X and Z are both closed in $(\mathbf{U}, \varphi)$ then $X \sqsubseteq Z$ if and only if $X \subseteq Z$, so on the closed sets it coincides with the subset partial ordering. Second, if $X.\gamma$ is a generator of $X.\varphi$, then for all Y with $X.\gamma \subseteq Y \subseteq X.\varphi$, Y also generates $X.\varphi$ and we have the reverse inclusion $X.\varphi \sqsubseteq Y \sqsubseteq X.\gamma$. Third, it can be shown [18] that if $(\mathbf{U}, \varphi)$ is a finite antimatroid, then $(\mathbf{U}, \sqsubseteq)$ defines a complete lattice, $\mathcal{L}$, called its **closure lattice** [18]. Again, we should properly use the notation $(2^{\mathbf{U}}, \sqsubseteq)$ since $\sqsubseteq$ is an ordering of the powerset, but this abuse of notation emphasizes the role of the base set $\mathbf{U}$. The lattices of Figures 1, 3 and 5 are illustrative of this $\sqsubseteq$ order.

The dash-dot lines in Figure 1 denote a few of the covering relationships between non-closed elements in different Boolean intervals. These covering relationships, which correspond to the ordering $X \sqsubseteq Z$, echo those of the closed subgraph sublattice. This pattern will be seen in all closure spaces where if X_1 is a generator of $Z_1 \sqsubseteq Z_2$, then there exists a generator X_2 of Z_2 such that $X_2 - X_1 = Z_2 - Z_1$, where neither X_1 nor X_2 need be minimal generators [18].

Finally, we observe that this closure is a T_0-topological closure, that is grounded (C0), finite union preserving (C4) and antimatroid (C6) closure. Thus, the order $\sqsubseteq$, restricted to singleton subsets, is precisely the *specialization order* $\sqsubseteq_s$ of the closure as described in [14].

When we used convex closure, φ_C, on the poset of Figure 4 we had many more closed subsets than were in Figure 1. Consequently, we have many fewer "non-trivial" generators as shown in Figure 5. Again, dashed lines demarcate subsets with the same closure. Observe that convex closure on any poset is an antimatroid, that $|\, Y.\gamma \,| \geq 2$ and that any finite set closed with respect to downset $\varphi_{\downarrow}$ or upset $\varphi_{\uparrow}$ will be closed with respect to φ_C.

The following proposition establishes that with posets the closure space ordering is conformant to the poset order. This, and the following 3 results, come from [18].

Proposition 1.4 *Let $(P, \varphi_{\downarrow})$ be the closure system given by a poset P. Then $x \leq y$ if and only if $x \sqsubseteq y$.*

Proof: Clearly, $x \leq y$ if and only if $x \in y.\varphi_{\downarrow}$. Thus, $x \leq y$ is equivalent to $\{y\} \cap \{x\}.\varphi_{\downarrow} \subseteq \{x\} \subseteq \{y\}.\varphi_{\downarrow}$ because $\{y\} \cap \{x\}.\varphi_{\downarrow} \neq \emptyset$ if and only if $x = y$. $\Box$

There are many interesting relationships between subset inclusion $\subseteq$ and the closure space (lattice) ordering $\sqsubseteq$ as determined by the closure operator φ.

Proposition 1.5 *If $P \cap X = \emptyset$ then*
(a) $X \sqsubseteq X \cup P$ if and only if $P \cap X.\varphi = \emptyset$,
(b) $X \cup P \sqsubseteq X$ if and only if $P \subseteq X.\varphi$.

Proof: (a) We have $X \sqsubseteq X \cup P$ iff $(X \cup P) \cap X.\varphi \subseteq X \subseteq (X \cup P).\varphi$, where the second set inclusion always holds, but the first is true iff $P \cap X.\varphi = \emptyset$.
(b) We have $X \cup P \sqsubseteq X$ iff $X \cap (X \cup P).\varphi \subseteq X \cup P \subseteq X.\varphi$, where the first inclusion always holds and the second one is true if and only if $P \subseteq X.\varphi$. $\Box$

Proposition 1.6 *Let $X \sqsubseteq Z$. If Z is closed, then X is closed.*

Proof: We have $Z \cap X.\varphi \subseteq X \subseteq Z = Z.\varphi$. Thus $X \subseteq Z$ and $X \subseteq Z.\varphi$ so $X \subseteq X \cap X.\varphi$ which as the intersection of closed sets must be closed. Since $X.\varphi$ is the smallest closed set containing X, $X = Z \cap X.\varphi = X.\varphi$. $\Box$
Sometimes the contrapositive, that is if $X \sqsubseteq Z$ and X is not closed then Z cannot be closed, is a more useful formulation.

In a discrete closure space, covering relationships take on a special importance, just as the existence of individual edges can be important in graph theory even though the transitive relationship is the path relation.

Proposition 1.7 *Let $(\mathbf{U}, \varphi)$ be an antimatroid and $Y, Z \subseteq \mathbf{U}$ subsets, Z closed. Then Z covers Y if and only if $Y = Z - p$ for some $p \in Z.\gamma$.*

Proof: Let $Y = Z - p$ for some $p \in Z.\gamma$. Then Y does not generate Z, hence Y is closed. Therefore, Z covers Y (because $\sqsubseteq$ coincides with set inclusion for closed sets). Conversely, let Z covers Y. By Prop. 1.6, Y is a maximal closed subset of Z different from Z. It follows that $Z.\gamma \not\subseteq Y$, hence there is $p \in Z.\gamma$ with $p \notin Y$. But then $Y \subseteq Z - p$. Since $Z - p$ is closed, we have $Y = Z - p$. $\Box$
It is worth verifying this relationship in Figure 1, as well as all other antimatroid closure

spaces. A more general relation between the generators of a closed set Z and the closed sets that it covers can be found in [11].

Since by Proposition 1.7, deletion of any generating (extreme) point, say b, from $Z = abcdef$ must create a new closed set $Y = acdef$. In [15], this repeated deletion of generating/extreme points is called "shelling" where they use it to define antimatroid behavior.

2 Transformations on Closure Systems

A transformation f of a closure system $\mathcal{C}$ into another $\mathcal{C}'$ is a function taking the power set of $\mathbf{U}$ into that of $\mathbf{U}'$. To simplify notation, we denote these functions by $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ instead of the more correct $2^{\mathbf{U}} \xrightarrow{f} 2^{\mathbf{U}'}$ and we call these functions "transformations" to emphasize their "set valued" nature.

A transformation $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ is said to be **monotone** if $\forall X, Y \subseteq \mathbf{U}$, $X \subseteq Y$ implies $X.f \subseteq Y.f$.

It is evident that each identity transformation is monotone and that the composition of monotone transformations is monotone. Of course, each point function $f : \mathbf{U} \to \mathbf{U}'$ determines a unique transformation $\mathbf{U} \xrightarrow{f^+} \mathbf{U}'$ given by lifting f, i.e., by putting $X.f^+ = \{y' \in \mathbf{U}' \mid \exists x \in X, x.f = y'\}$.

2.1 Complete, Continuous and Closed Transformations

A transformation f simply maps subsets of $\mathbf{U}$ into subsets of $\mathbf{U}'$. The real focus of this paper is what happens to the closure properties of $\mathbf{U}$ under f. To emphasize this, we will usually denote a transformation by $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$. The question we address in this section is "can we characterize those transformations which preserve the closure structure?", that is for which $Y.\varphi.f = Y.f\varphi'$.

The most commonly accepted definition of "continuity" in discrete closure spaces is that a transformation $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ is **continuous** if $\forall Y \subseteq \mathbf{U}$, $Y.\varphi.f \subseteq Y.f.\varphi'$ [23]. This is the definition we use in this paper.

An important property of "continuity" is that the inverse image of closed sets ought to be closed. Unfortunately, this is not a property of continuity, as defined above. Consider $\mathbf{U} = \{a, b\}$ and $\mathbf{U}' = \{x, y, z\}$, whose closed sets are $\mathcal{C}_\varphi = \{ab\}$ and $\mathcal{C}_{\varphi'} = \{xy, xyz\}$. The transformation $\emptyset.f = \emptyset$, $a.f = xy$, $b.f = xyz$, and $ab.f = x$ is easily shown to be continuous, but the inverse image of the only non-trivial closed set in $\mathbf{U}'$, $xy.f^{-1} = a$ which is not closed in $\mathbf{U}$. (We see this example again in Figure 8.) Somehow the notion of "continuity" must be strengthened.

A transformation $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ is said to be **complete** if $\forall X \in \mathbf{U}$, $X.f$ closed in $(\mathbf{U}', \varphi')$ implies $X.\varphi.f = X.f$.

In a discrete space, "completeness" has some of the characteristics more commonly associated with "continuity". In particular, it provides an analog to the notion that "the inverse images of closed sets are closed". To get a sense of why we might want a transformation to be "complete" consider the simple transformation f which maps a linear order on the 3 points, a, b, and c shown in Figure 7 onto a linear order on just 2 points a', c', where $a.f = a', c.f = c'$ and $b.f$ can be either a' or c'. This f is clearly "order preserving" and

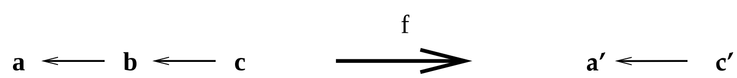


Figure 7: A "complete" transformation

could be regarded as an epitome of a "well-behaved" discrete function. The subset $a'c'$ is closed with respect to any of the closure operators that we normally associate with such partial orders, while $ac \in \{a'c'\}.f^{-1}$ is closed with respect to none of them. Clearly, we can't require every pre-image set to be closed; but by insisting that whenever the image is closed, the closure of a pre-image set must also map onto this image set, we believe we have captured an essential property. But, complete transformations need not be continuous in the sense given above.

Proposition 2.1 *A transformation* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *that is monotone and continuous is complete.*

Proof: Let $Y.f$ be closed in $\mathbf{U}'$. Because f is continuous $Y.\varphi.f \subseteq Y.f.\varphi' = Y.f$, since $Y.f$ is closed. By monotonicity, $Y.f \subseteq Y.\varphi.f$, so $Y.\varphi.f = Y.f$. $\square$

We will say a transformation $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ is **closed** if f takes closed sets in $(\mathbf{U}, \varphi)$ onto closed sets in $(\mathbf{U}', \varphi')$.

Clearly, the identity transformation of a closure system is continuous and complete.

The transformation properties of monotonicity, continuity and closedness are independent, as shown by the following Figure 8. Here $\mathbf{U} = \{a, b\}$, $\mathbf{U}' = \{x, y, z\}$, $\mathcal{C}_\varphi = \{ab\}$, $\mathcal{C}_{\varphi'} = \{xy, xyz\}$. In the first three cases each f_i satisfies one property, but not the other two. The f_4 transformation is monotone and closed and complete (as can be easily verified), but it is not continuous. So Proposition 2.1 can have no converse.

It is obvious that the composition of closed transformations is a closed transformation. On the other hand, the same is not true for continuous transformations as the following example shows:

Let $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$, $(\mathbf{U}', \varphi') \xrightarrow{g} (\mathbf{U}'', \varphi'')$ be the transformations with $\mathbf{U} = \{a, b\}$, $\emptyset.\varphi = \emptyset$, $A.\varphi = U$ if $\emptyset \neq A \subseteq U$, $\mathbf{U}' = \{x, y, z\}$, $A.\varphi' = \mathbf{U}'$ if $x \in A \subseteq \mathbf{U}'$ and $A.\varphi' = A$

$Y \in \mathbf{U}$	$Y.f_1$	$Y.f_2$	$Y.f_3$	$Y.f_4$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
a	xy	y	xy	x
b	xyz	y	xyz	y
ab	x	yz	xy	xyz
	$Cont$	$\neg Cont$	$\neg Cont$	$\neg Cont$
	$\neg Mono$	$Mono$	$\neg Mono$	$Mono$
	$\neg Clos$	$\neg Clos$	$Clos$	$Clos$

Figure 8: Four transformations, $\mathbf{U} \xrightarrow{f_k} \mathbf{U}'$

if $x \notin A \subseteq \mathbf{U}'$, $\mathbf{U}'' = \{p, q\}$, $\emptyset.\varphi'' = \emptyset$, $\{p\}.\varphi'' = \{p\}$, $\{q\}.\varphi'' = \mathbf{U}''.\varphi'' = \mathbf{U}''$, $\emptyset.f = \emptyset$, $\{a\}.f = \{x\}$, $\{b\}.f = \mathbf{U}.f = \{x, y\}$, $\emptyset.g = \emptyset$, $\{x.y\}.g = \mathbf{U}''$ and $A.g = \{p\}$ whenever $A \subseteq \mathbf{U}'$ is a subset with $\emptyset \neq A \neq \{x, y\}$. Then both f and g are continuous but $f \cdot g$ is not because $\{a\}.\varphi.(f \cdot g) = \mathbf{U}''$ while $\{a\}.(f \cdot g).\varphi'' = \{p\}$.

But we have

Proposition 2.2 *Let* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$, $(\mathbf{U}', \varphi') \xrightarrow{g} (\mathbf{U}'', \varphi'')$ *be transformations and let* g *be monotone. If both* f *and* g *are continuous, then so is* $\mathbf{U} \xrightarrow{f \cdot g} \mathbf{U}''$.

Proof: We have $X.\varphi.f \subseteq X.f.\varphi'$ for any $X \in \mathbf{U}$ and $Y.\varphi'.g \subseteq Y.g.\varphi''$ for any $Y \in \mathbf{U}'$. Consequently, as g is monotone, $X.\varphi.f.g \subseteq X.f.\varphi'.g \subseteq X.f.g.\varphi''$. This means that $f \cdot g$ is continuous. □

Proposition 2.3 *A monotone transformation* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is closed if and only if* $\forall X \subseteq \mathbf{U},\ X.f.\varphi' \subseteq X.\varphi.f$.

Proof: Let f be closed. By monotonicity, $X \subseteq X.\varphi$ implies $X.f \subseteq X.\varphi.f$. But, because $X.\varphi$ is closed and f is closed, $X.f.\varphi' \subseteq X.\varphi.f$
Conversely, let all subsets $X \subseteq \mathbf{U}$ fulfill $X.f.\varphi' \subseteq X.\varphi.f$ and let X be a closed subset of $(\mathbf{U}, \varphi)$. Then $X.f.\varphi' \subseteq X.f$. But, readily $X.f \subseteq X.f.\varphi'$, so equality holds. □

Theorem 2.4 *A monotone transformation* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is closed and continuous if and only if, for all* $X \subseteq \mathbf{U}$, $X.\varphi.f = X.f.\varphi'$.

Proof: Proposition 2.3 and the definition of continuity □

Theorem 2.4 provides necessary and sufficient conditions on a monotone transformation f to ensure that f commutes with the closure operator φ as in Figure 9.

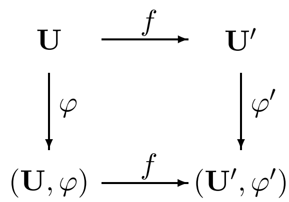


Figure 9: f regarded as a closure system transformation

2.2 Homomorphisms of Posets

Recall that $f : P \to P'$ is a **homomorphism** of the poset P into P' if $x \leq y$ in P implies $f(x) \leq f(y)$ in P'.[4] Since an anti-chain can be the pre-image of any ordered set under a homomorphism, it would be an exaggeration to call them "structure preserving". Often one wants the structure of the domain to have some resemblance to its image. For this reason some authors prefer "strong homomorphisms" where a homomorphism is called **strong** if in addition $x' \leq y'$ in P' implies $\exists x, y \in P$ such that $f(x) = x', f(y) = y'$ and $x \leq y$.

Proposition 2.5 *A function $f : (P, \leq) \to (P', \leq')$ is a homomorphism if and only if the lifted function $(P, \varphi) \xrightarrow{f^+} (P', \varphi')$ is continuous with respect to a path closure.*

Proof: Let f be a homomorphism and let $Y \subseteq P$. Let $x' \in Y.\varphi_{\downarrow}.f^+$. Then $\exists x, y, \in Y.\varphi_{\downarrow}$ where $x \leq y$ and $x' = f(x)$. Consequently, $f(x) \leq f(y)$. Therefore, $x' = f(x) \in Y.f^+.\varphi_{\downarrow}$. So $Y.\varphi_{\downarrow}.f^+ \subseteq Y.f^+.\varphi_{\downarrow}$.
Conversely, assume that f^+ is continuous and $x \leq y$. Then $x \in y.\varphi_{\downarrow}$ and $x.\varphi_{\downarrow} \subseteq y.\varphi_{\downarrow}$. $\{x\}.f \subseteq \{x\}.\varphi_{\downarrow}.f^+ \subseteq \{y\}.\varphi\downarrow.f^+ \subseteq \{y\}.f^+.\varphi_{\downarrow}$. Thus $f(x) \leq f(y)$.
The proof when the closure is φ_C or $\varphi_{\uparrow}$ is similar. □

In an early application of this principle [17] it was shown that if $f : G \to G'$ was any graph homomorphism, then G' would be acyclic if and only if the inverse image of every point in G' was a convex (in the sense of φ_C of Section 1.3) set in G.

But, poset homomorphisms need not be closed. In Figure 10, $\{a, b, d\}$ is closed (assuming

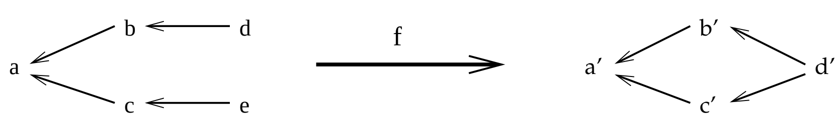


Figure 10: A homomorphism f that is not closed.

[4]Note that we use the traditional prefix notation $f : \mathbf{U} \to \mathbf{U}'$ for point functions on $\mathbf{U}$, and $\mathbf{U} \xrightarrow{f^+} \mathbf{U}'$ for transformations between the power sets.

$\varphi_{\downarrow}$ closure), but $\{a', b', d'\}$ is not. Even strong homomorphisms need not be closed, as shown by Figure 11, because $\{b_2\}$ is closed but $\{b_2\}.g = \{b'\}$ is not. However, we can show

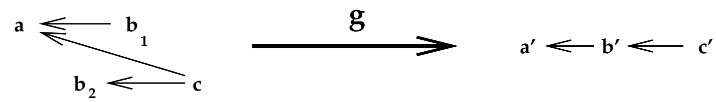


Figure 11: A strong homomorphism g that is not closed.

Proposition 2.6 *Let $g : (P, \leq) \rightarrow (P', \leq)$ be a surjective homomorphism. If $(P, \varphi) \xrightarrow{g^+} (P, \varphi')$ is closed with respect to a path closure, then g is a strong homomorphism.*

Proof: Let g^+ be closed, let $x' \leq y'$ in P' and let $\{y'\}.g^{-1} = Y$. Then $Y.g^+ = \{y'\}$, so $x' \in Y.g^+.\varphi_{\downarrow} \subseteq Y.\varphi_{\downarrow}.g^+$ implying there exists $y_k \in Y$ and $x_i \in Y.\varphi_{\downarrow}$ such that $x_i \leq y_k$ where $x_i.g = x'$ and by definition of Y, $y_k.g = y'$. So g^+ is strong. □

Combining Prop. 2.5 with Prop. 2.6 we readily have that if g^+ is closed, then g is a homomorphism so g^+ must also be continuous. But, only in this special case where we have lifted functions over poset closures need this be true. It is not a general implication. Similarly, in the special case of poset homomorphism, the completeness property can be shown to be completely equivalent to continuity.

2.3 Intersection of Transformations

Given a family f_i, $i \in I$, of numeric functions (with the same domain and range), one can define the operators $\sum_{i \in I} f_i$ of sum and $\prod_{i \in I} f_i$ of product of this family in a pointwise fashion; that is $\sum_{i \in I} f_i$ and $\prod_{i \in I} f_i$ are defined $\forall x$ to be the sum $\sum_{i \in I} f_i(x)$ and the product $\prod_{i \in I} f_i(x)$, respectively. Since domains of our transformations are power sets, we may define the intersection, $\bigcap_{i \in I} f_i$, or meet, of a family $\mathbf{U} \xrightarrow{f_i} \mathbf{U}'$, $i \in I$, of transformations in a "set-wise" fashion, i.e., as $Y.\bigcap_{i \in I} f_i = \bigcap_{i \in I} Y.f_i$ for each $Y \subseteq \mathbf{U}$. Earlier we had observed that $\varphi_C = \varphi_{\downarrow} \cap \varphi_{\uparrow}$.

Of course, the intersection of monotone transformations is always monotone (because, given a family $\mathbf{U} \xrightarrow{f_i} \mathbf{U}'$, $i \in I$, of monotone transformations and subsets $Y\ Z \subseteq \mathbf{U}$, $Y \subseteq Z$, we have $Y.\bigcap_{i \in I} f_i = \bigcap_{i \in I} Y.f_i \subseteq \bigcap_{i \in I} Z.f_i = Z.\bigcap_{i \in I} f_i$). On the other hand, neither continuity nor completeness of transformations is closed under (i.e., preserved by) intersections as the following two examples show:

1) Let $\mathbf{U} = \{a, b\}$ and $\mathbf{U}' = \{x, y, z\}$. Let $\{a\}.\varphi = \mathbf{U}$ and $X.\varphi = X$ whenever $\{a\} \neq X \subseteq \mathbf{U}$. Let, $\{x\}.\varphi' = \{x, z\}$, $\{y\}.\varphi' = \{y, z\}$, $\{x, y\}.\varphi' = \mathbf{U}$ and $X.\varphi = X$ for all the other subsets $X \subseteq \mathbf{U}$. Let $(\mathbf{U}, \varphi) \xrightarrow{f,g} (\mathbf{U}', \varphi')$ be the transformations given by $\{a\}.f = \{x\}$, $\{b\}.f = \{z\}$, $\emptyset.f = \emptyset$, $\mathbf{U}.f = \{x, z\}$, and $\{a\}.g = \{y\}$, $\{b\}.g = \{z\}$, $\emptyset.g = \emptyset$, $\mathbf{U}.g = \{y, z\}$.

Then both f and g are continuous, $\{a\}.\varphi.(f \cap g) = \{z\}$ but $\{a\}.(f \cap g).\varphi' = \emptyset$. So, $f \cap g$ is not continuous.

2) Let $\mathbf{U} = \{a, b, c\}$ and $\mathbf{U}' = \{x, y, z, t, u\}$. Let $\{a\}.\varphi = \{a, b\}.\varphi = \{a, c\}.\varphi = \mathbf{U}$ and $X.\varphi = X$ whenever $\{a\} \neq X \subseteq \mathbf{U}$. Let, for each subset $Y \subseteq \mathbf{U}'$, $Y.\varphi' = Y \cup A$ where $A = \{x, t\}$ if $Y \cap \{y, z\} = \{y\}$, $A = \{x, u\}$ if $Y \cap \{y, z\} = \{z\}$, $A = \{x, t, u\}$ if $Y \cup \{y, z\} = \{y, z\}$, and $A = \emptyset$ if $Y \cap \{y, z\} = \emptyset$. Finally, let $(\mathbf{U}, \varphi) \xrightarrow{f,g} (\mathbf{U}', \varphi')$ be the transformations given by $\{b\}.f = \{x\}$, $\{a\}.f = \{a, b\}.f = \{x, y\}$, $\emptyset.f = \emptyset$, $X.f = \mathbf{U}'$ otherwise, and $\{b\}.g = \{x\}$, $\{a\}.g = \{a, b\}.g = \{x, z\}$, $\emptyset.g = \emptyset$, $X.g = \mathbf{U}'$ otherwise. Then both f and g are complete, $\{a, b\}.(f \cap g) = \{x\}$ is closed but $\{a, b\}.\varphi.(f \cap g) = \mathbf{U}.(f \cap g) = \mathbf{U}'$. So, $f \cap g$ is not complete.

Fortunately, for closedness the situation is favorable:

Proposition 2.7 *If $(\mathbf{U}, \varphi) \xrightarrow{f_i} (\mathbf{U}', \varphi')$, $i \in I$ is a family of closed transformations, then $\bigcap_{i \in I} f_i$ is closed too.*

Proof: Let f_i, $i \in I$, be closed and let $Y \subseteq \mathbf{U}$ be a subset. Then $Y.\varphi.\bigcap_{i \in I} f_i = \bigcap_{i \in I} Y.\varphi.f_i \supseteq \bigcap_{i \in I} Y.f_i.\varphi' \supseteq (\bigcap_{i \in I} Y.f_i).\varphi' = Y.(\bigcap_{i \in I} f_i).\varphi'$. $\square$

The transformations from $\mathbf{U}$ into $\mathbf{U}'$ may be identified with the disjoint unions $\coprod_{Y \subseteq U} Z'_Y$ $= \bigcup_{Y \subseteq U} (\{Y\} \times Z'_Y)$ where $Z'_Y \subseteq \mathbf{U}'$ for all $Y \subseteq \mathbf{U}$. Namely, every such a disjoint union gives the transformation $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ with $Y.f = Z'_Y$ for all $Y \subseteq U$ and conversely. Given a pair f, g of transformations from $\mathbf{U}$ into $\mathbf{U}'$, we clearly have $f \subseteq g$ if and only if $Y.f \subseteq Y.g$ for each subset $Y \subseteq \mathbf{U}$. The smallest (with respect to the set inclusion) of all transformations from $\mathbf{U}$ into $\mathbf{U}'$ is the transformation $\mathbf{U} \xrightarrow{\mathcal{O}} \mathbf{U}'$ given by $Y.\mathcal{O} = \emptyset$ for each $Y \subseteq \mathbf{U}$, i.e., $\mathcal{O} = \coprod_{Y \subseteq U} Y.f$ where $Y.f = \emptyset$ for each $Y \subseteq \mathbf{U}$. Thus, we have $\mathcal{O} = \emptyset$. On the other hand, the greatest of all transformations from $\mathbf{U}$ into $\mathbf{U}'$ is the transformation $\mathbf{U} \xrightarrow{\mathcal{I}} \mathbf{U}'$ given by $Y.\mathcal{I} = \mathbf{U}'$ for each $Y \subseteq \mathbf{U}$, i.e., $\mathcal{I} = \coprod_{Y \subseteq U} Y.f$ where $Y.f = \mathbf{U}'$ for each $Y \subseteq \mathbf{U}$. Of course, both $\mathcal{O}$ and $\mathcal{I}$ are monotone transformations. Note that the points of $\mathcal{I}$ coincide with the transformations $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ for which there exists a subset $Y_0 \subseteq \mathbf{U}$ such that $Y_0.f$ is a singleton and $Y.f = \emptyset$ for all $Y \subseteq U$, $Y \neq Y_0$. Obviously, each transformation from $\mathbf{U}$ to $\mathbf{U}'$ may be obtained as the disjoint union of some of these "atomic" transformations.

Given a pair $\mathcal{C}_1$, $\mathcal{C}_2$ of closure systems, we denote by $[\mathcal{C}_1, \mathcal{C}_2]$ the set of all closed transformations from $\mathcal{C}_1$ into $\mathcal{C}_2$. Clearly, the transformation $\mathcal{I}$ (from $\mathcal{C}_1$ into $\mathcal{C}_2$) belongs to $[\mathcal{C}_1, \mathcal{C}_2]$. Now, Proposition 2.7 results in

Proposition 2.8 *Let $\mathcal{C}_1, \mathcal{C}_2$ be closure systems. Then $[\mathcal{C}_1, \mathcal{C}_2]$ is a closure system on the universe $\mathcal{C}_1 \xrightarrow{\mathcal{I}} \mathcal{C}_2$.*

3 Categories of Closure Systems with Transformations as Morphisms

For the categorical terminology used see [1] and [21].

The category of grounded closure systems and continuous (point) functions has been studied by many authors, see e.g. [3] and [6]. Since this category is not cartesian closed, in [3] the authors construct its cartesian closed extension by adding new objects and morphisms between them. In our approach, instead of point functions, transformations are used as morphisms between closure systems. As described in Section 2, all point functions can be simply lifted to become transformations, so that transformations can be considered to be extensions of point functions. Using this kind of extension, we will find a cartesian closed category whose objects are just the closure systems.

All categories considered in this section will have all closure systems as the class of objects and certain transformations as morphisms. Thus, these categories may be given by just specifying the transformations that are their morphisms. By results of Section 2, all monotone continuous transformations constitute a category - we denote it by **MCont** - and also all monotone closed transformations constitute a category - we denote it by **MClo**. Clearly, both **MCont** and **MClo** are concrete categories over **Set** with the underlying functor F given by $F(\mathbf{U}, \varphi) = 2^{\mathbf{U}}$ for objects and $Ff = f$ for morphisms. The closure system $\mathcal{C}_0$ with the empty universe is the unique **terminal object** in both **MCont** and **MClo** because the transformation f defined by $X.f = \emptyset, \forall X \in \mathcal{C}$ is a unique monotone, continuous and closed transformation from $\mathcal{C}$ into $\mathcal{C}_0$. Observe that in the category **Set**, the empty set is the initial object and singletons are terminal objects [21]. The difference is that, in the category of closure systems, morphisms act on power sets of universes and not on the universes themselves.

Let $\mathcal{C}_i$, $i \in I$ (I a set), be a family of closure systems on the base sets $\mathbf{U}_i$. We let the disjoint union $\coprod_{i\in I} \mathbf{U}_i$ be the base set of the **direct product** $\prod_{i\in I} \mathcal{C}_i$ whose closed sets are defined to be just the sets of the form $\coprod_{i\in I} X_i$ where $X_i \in \mathcal{C}_i$ are closed sets for each $i \in I$. Thus, if $\prod_{i\in I} \mathcal{C}_i = (\coprod_{i\in I} \mathbf{U}_i, \varphi)$ and $\mathcal{C}_i = (\mathbf{U}_i, \varphi_i)$, we have $\varphi_i = \varphi|_{U_i}$ (for each $i \in I$). If $I = \{i_1, i_2\}$, we write $X_{i_1} \uplus X_{i_2}$ and $\mathcal{C}_{i_1} \times \mathcal{C}_{i_2}$ instead of $\coprod_{i\in I} X_i$ and $\prod_{i\in I} \mathcal{C}_i$, respectively.

Proposition 3.1 *The direct product is a concrete product in both* **MCont** *and* **MClo***.*

Proof: Let $\mathcal{C}_i$, $i \in I$, be a family of closure systems. Of course, if $I = \emptyset$, then $\prod_{i\in I} \mathcal{C}_i$ is the terminal object. So, we may suppose that $I \neq \emptyset$. First, we will show that $\prod_{i\in I} \mathcal{C}_i$ is a closure system. Let Z_j, $j \in J$, be closed sets in $\prod_{i\in I} \mathcal{C}_i$. Then, for each $j \in J$, $Z_j = \coprod_{i\in I} X_i^{(j)}$ for some $X_i^{(j)} \in \mathbf{U}_i$, $i \in I$. We have $\bigcap_{j\in J} Z_j = \bigcap_{j\in j} \coprod_{i\in I} X_i^{(j)} = \coprod_{i\in I} \bigcap_{j\in J} X_i^{(j)}$, hence $\bigcap_{j\in J} Z_j$ is closed in $\prod_{i\in I} \mathcal{C}_i$. We can now define **projection** morphisms $\prod_{i\in I} \mathcal{C}_i \xrightarrow{\pi_i} \mathcal{C}_i$, $i \in I$, by $Z.\pi_i = Z \cap \mathbf{U}_i$. Clearly, π_i is a (surjective) monotone transformation for each $i \in I$. But π_i is also continuous for each $i \in I$ because, given a subset $Z \subseteq \coprod_{i\in I} U_i$, we have $Z.\varphi.\pi_i = Z.\varphi \cap U_i = Z.\pi_i.\varphi_i$.

Now, let $\mathcal{C} = (\mathbf{U}, \psi)$ be a closure system and let $\mathcal{C} \xrightarrow{f_i} \mathcal{C}_i$, $i \in I$, be morphism in **MCont**. Let $\mathcal{C} \xrightarrow{h} \prod_{i\in I} \mathcal{C}_i$ be the transformation given by $\forall Z \in \mathcal{C}: \ Z.h = \coprod_{i\in I} Z.f_i \in \prod_{i\in I} \mathcal{C}_i$. Clearly, $f_i = h \cdot \pi_i$ for each $i \in I$ and h is a unique transformation having this property. It is obvious that h is monotone (because f_i, $i \in I$, are monotone). For every subset $Z \subseteq \mathcal{C}$ we have $Z.\psi.h = \coprod_{i\in I} Z.\psi.f_i \subseteq \coprod_{i\in I} Z.f_i\varphi_i \subseteq (\coprod_{i\in I} Z.f_i).\varphi_i = Z.h.\varphi_i$. Therefore, h is continuous. We have shown that $\prod_{i\in I} \mathcal{C}_i$ is a product in **MCont**.
Obviously, given a family $\mathcal{C}_i$, $i \in I$ of closure systems, the projections $\prod_{i\in I} \mathcal{C}_i \xrightarrow{\pi_i} \mathcal{C}_i$ are closed for all $i \in I$. Further, let $\mathcal{C} = (\mathbf{U}, \psi)$ be a closure system and $\mathcal{C} \xrightarrow{f_i} \mathcal{C}_i$, $i \in I$, be closed morphisms. Let $Z \subseteq \mathcal{C}$ be a closed subset. Then $Z.h = \coprod_{i\in I} Z.f_i$ and, since f_i $(i \in I)$ are closed, $Z.f_i$ $(i \in I)$ are closed too. Consequently, $Z.h$ is closed. Thus, with respect to the above considerations, $\prod_{i\in I} \mathcal{C}_i$ is a product in **MClo**.
It is clear that $\prod_{i\in I} \mathcal{C}_i$ is a concrete product in each of the categories **MCont** and **MClo** because $\mathrm{F}(\prod_{i\in I} \mathcal{C}_i) = 2^{\coprod_{i\in I} \mathcal{C}_i} \cong \prod_{i\in I} 2^{\mathcal{C}_i} = \prod_{i\in I} F(\mathcal{C}_i)$. $\square$

Since the direct product $\prod_{i\in I} \mathcal{C}_i$ in **MCont** and **MClo** is a concrete product over **Set**, it is clear now why its universe is a coproduct of the universes of the individual closure systems $\mathcal{C}_i$, $i \in I$.

Of course, it immediately follows from Proposition 3.1 that $\prod_{i\in I} \mathcal{C}_i$ is a concrete product also in **MCont** $\cap$ **MClo**, i.e., in the category of closure systems with monotone continuous and closed transformations as morphisms (see Theorem 2.4). While this category is a subcategory of **MClo**, there is a supercategory **Clo** of **MClo** obtained by omitting the requirement of monotonicity of the morphisms. Thus, **Clo** has just the closed transformations as morphisms. It is clear from the proof of Proposition 3.1 that the direct product $\prod_{i\in I} \mathcal{C}_i$ is a concrete product also in **Clo**. The category **Clo** is especially interesting because we will show now that it is a so-called cartesian closed category.

Recall that a category **C** is **cartesian closed** [13, 21] if it has finite products and for any two objects $A, B \in \mathbf{C}$ there exists an object B^A in **C** and a morphism $B^A \times A \xrightarrow{ev} B$ with the property that for each $C \times A \xrightarrow{f} B$ there exists a unique morphism $C \xrightarrow{g} B^A$ such that $(g \times id_a) \cdot ev = f$, i.e., such that the diagram on Fig. 12 commutes. The morphism ev is then said to be the evaluation morphism and the object B^A is called the *exponential object*.

Thus, in addition to finite multiplication, cartesian closed categories have also the operation of exponentiation on objects and these two operations are well behaved, i.e., they fulfill analogies of the usual laws which are valid for multiplication and exponentiation of natural numbers (the most important of them is the so-called first exponential law $(A^B)^C \cong A^{B\times C}$). Because of the existence of this natural arithmetic, cartesian closed categories have many applications in different branches of mathematics and informatics. For example, cartesian closed categories play an especially important role in logic because it is well known that

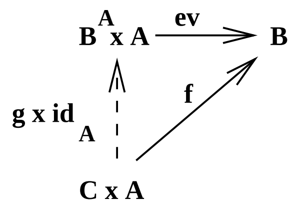


Figure 12: Exponential diagram.

they form models of the so-called typed lambda-calculus, which is an important formal programming language.

Theorem 3.2 *The category* **ClosureSys** *is cartesian closed with exponential objects given by* $\mathcal{C}_2^{\mathcal{C}_1} = [\mathcal{C}_1, \mathcal{C}_2]$ *for all closure systems* $\mathcal{C}_1,\ \mathcal{C}_2$.

Proof: We have seen that **ClosureSys** has (not only) finite products given by direct products. Let $\mathcal{C}_1,\ \mathcal{C}_2$ be closure systems and let $[\mathcal{C}_1, \mathcal{C}_2] \times \mathcal{C}_1 \xrightarrow{ev} \mathcal{C}_2$ be the transformation given by $(h \uplus Y).ev = Y.h$ whenever $h \subseteq |[\mathcal{C}_1, \mathcal{C}_2]|$ (i.e., $\mathcal{C}_1 \xrightarrow{h} \mathcal{C}_2$ is a transformation) and $Y \subseteq \mathcal{C}_2$. If h is a closed transformation and Y is a closed subset, then $Y.h$ is closed too, which means that ev is a closed transformation. Let $\mathcal{D}$ be a closure system and $\mathcal{D} \times \mathcal{C}_1 \xrightarrow{f} \mathcal{C}_2$ be a closed transformation. Let $\mathcal{D} \xrightarrow{g} [\mathcal{C}_1, \mathcal{C}_2]$ be the transformation given by $Y.(Z.g) = (Z \uplus Y).f$ for all $Z \subseteq |\mathcal{D}|$ and all $Y \subseteq |\mathcal{C}_1|$. Then $(Z \uplus Y).(g \times id_{\mathcal{C}_1}).ev = (Z.g \uplus Y).ev = Y.(Z.g) = (Z \uplus Y).f$. Thus, we have found a unique transformation g for which the diagram from the definition of a cartesian closed category commutes. To complete the proof, we are to show that g is closed. To this account, let $Z \subseteq |\mathcal{D}|$ be a closed subset. Then, whenever $Y \subseteq |\mathcal{C}_1|$ is a closed subset, also $Y.(Z.g) \subseteq |\mathcal{C}_2|$ is closed because $Y.(Z.g) = (Z \uplus Y).f$ where f is closed. $\square$

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