Transformations of Discrete Closure Systems

John L. Pfaltz
Dept. of Computer Science, Univ. of Virginia
jlp@virginia.edu

Josef Šlapal*
Dept. of Mathematics, Brno Univ. of Technology
slapal@fme.vutbr.cz

May 13, 2012

Abstract

Discrete systems such as sets, monoids, groups are familiar categories. The internal structure of the latter two is defined by an algebraic operator. In this paper we concentrate on discrete systems that are characterized by unary operators; these include choice operators σ, encountered in economics and social theory, and closure operators φ, encountered in discrete geometry and data mining. Because, for many arbitrary operators α, it is easy to induce a closure structure on the base set, closure operators play a central role in discrete systems.

Our primary interest is in functions f that map power sets $2^{\mathbf{U}}$ into power sets $2^{\mathbf{U}'}$, which are called transformations. Functions over continuous domains are usually characterized in terms of open sets. When the domains are discrete, closed sets seem more appropriate. In particular, we consider monotone transformations which are "continuous", or "closed". These can be used to establish criteria for asserting that "the closure of a transformed image under f is equal to the transformed image of the closure".

Finally, we show that the categories **MCont** and **MClo** of closure systems with morphisms given by the monotone continuous transformations and monotone closed transformations respectively have concrete direct products. And the supercategory **Clo** of **MClo** whose morphisms are just the closed transformations is shown to be cartesian closed.

Keywords: closure; choice; operator; continuous; category; function

*The second author acknowledges a partial financial support from the Ministry of Education of the Czech Republic, project no. MSM 0021630518.

1 Discrete Systems

By a discrete system, $\mathcal{S}$, we mean a set of elements, points, or other phenomena which we will generically call our *universe*, denoted by $\mathbf{U}$, together with a structure defined by an arbitrary operator which we denote by α.[1] Thus $\mathcal{S} = (\mathbf{U}, \alpha)$. Operators may be unary, binary or n-ary; although in this paper we concentrate on unary operators. Individual points of $\mathbf{U}$ will be denoted by lower case letters: $a, b, ..., p, q, ... \in \mathbf{U}$. Discrete systems are typically finite, but need not be. The integers with, say the successor operator, constitute a discrete system. The defining characteristic is that no limit process is possible in a discrete system; that is, there are no accumulation points.

If a finite set of points of $\mathbf{U}$ is given by listing them, we may not write commas between the letters denoting these individual points and, moreover, we often omit the set-denoting parentheses. So, for example, we may write $\{abcd\}$ or even $abcd$ instead of $\{a, b, c, d\}$. By $2^{\mathbf{U}}$, we mean the power set on $\mathbf{U}$, *i.e.*, the collection of all subsets of $\mathbf{U}$. Elements of $2^{\mathbf{U}}$ we will denote by upper case letters: S, T, X, Y, Z.

A transformation, f, mapping one space into another should preserve at least some of the structure of the space. But, what is its structure? One way of describing a structure is by operators defined on the space. The transformation should preserve the operator behavior. The homomorphisms of discrete algebraic systems are a classic example [3, 41]. In this section we review several unary operators that have been used to describe a variety of discrete spaces. Our goal is to convey a sense of how mathematically rich the domain of these transformations can be. The nature of such transformations, which we consider in sections 2 and 3, has been less well studied; most of these results are original.

1.1 Operators

An operator α in a system $\mathcal{S}$ is a function $2^{\mathbf{U}} \rightarrow 2^{\mathbf{U}}$ which takes subsets of $\mathbf{U}$ onto other subsets of $\mathbf{U}$. Operators can have many properties; for example, α can be **monotone,** that is for all $X, Y \subseteq \mathbf{U}$, $X \subseteq Y$ implies $X.\alpha \subseteq Y.\alpha$. We use suffix notation to denote operators.

Two important properties are whether α is **contractive**, that is $Y.\alpha \subseteq Y$ or **expansive**, that is $Y \subseteq Y.\alpha$. Contractive operators may be generically called "choice" operators and have been extensively studied in the social sciences. Expansive operators are often "closure" operators, and will be the focus of this paper.

An operator, α, is said to be **path independent** if for all $X, Y \subset \mathbf{U}$,

$$(X \cup Y).\alpha = (X.\alpha \cup Y.\alpha).\alpha. \tag{1}$$

[1]Specific operators will later be denoted by σ, φ, and γ.

1.2 Choice Operators

The theory of choice functions was initially developed by economists, such as Arrow and Sen [4, 39], and further expanded by mathematicians, such as Aizerman and Malishevski [2, 22]. In this section we will summarize these, and later, results for the purpose of contrasting choice operators with closure operators in the next subsection. Closure will be the primary mechanism by which we describe general transformations. However, closure and choice can be closely linked [20].

Given a set Y of possible alternatives, a **choice operator** σ chooses one, or more, of these alternatives. Thus for all non-empty $Y \subseteq \mathbf{U}$, $Y.\sigma \neq \emptyset$ and $Y.\sigma \subseteq Y$. Consequently, σ is a contractive operator and one may call all such contractive operators "choice" operators.

The goal of economists has been to reconcile choice behaviors with rational behavior. Path independence, (1), was introduced by Plott [35] to capture a key element of rational choice, namely that choice should be independent of the order in which alternatives are considered. Thus, regardless of whether the alternatives X and Y are presented serially or altogether as $X \cup Y$, the resultant choice should be the same [22].

Path independent choice operators have a number of elegant properties. For example, one can define a binary operator $+$ on $\mathbf{U}$ by $X + Y \equiv X.\sigma \cup Y.\sigma$. If σ is path independent, then $(2^{\mathbf{U}}, +)$ is a commutative semi-group [22, 24]. Similarly, we might order a collection of chosen sets $\mathcal{C} = \{C_0, \ldots C_n\}$ of $2^{\mathbf{U}}$ by $C_i \leq C_k$ if and only if $(C_i \cup C_k).\sigma = C_k$. Readily $\mathcal{C}$, so ordered, is a join semi-lattice. If σ is path independent, then $\mathcal{C}$ is a convex geometry as described in Section 1.4 [17, 20].

1.3 Closure Operators

A **closure system**, $\mathcal{S}$, is a pair $(\mathbf{U}, \mathcal{C})$ where $\mathcal{C}$ is any family of subsets of $\mathbf{U}$, including $\mathbf{U}$ itself, which is closed under arbitrary intersections. Subsets in $\mathcal{C}$ are said to be **closed**. Since the universe $\mathbf{U}$ is contained in $\mathcal{C}$, $\mathcal{C}$ itself will be often be regarded as the closure system. In this case, $\mathcal{C}$ is said to be a closure system on $\mathbf{U}$ and $\mathbf{U}$ is said to be the base universe of $\mathcal{C}$. For example, if $\mathbf{U} = \{a, b, c, d, e\}$ then the collection of closed sets

$$\mathcal{C}_{\mathbf{U}} = \{\emptyset, \{a\}, \{b\}, \{ab\}, \{bd\}, \{abc\}, \{abd\}, \{abce\}, \{abcde\}, \{abcdef\}\} \tag{2}$$

is a closure system.

A closure system, $\mathcal{S}$, can equivalently be defined as a pair $(\mathbf{U}, \varphi)$, where φ is a **closure operator** on $\mathbf{U}$, *i.e.*, a self-mapping of $2^{\mathbf{U}}$, satisfying the following three axioms: For all $Y, Z \subseteq \mathbf{U}$,

C1: $Y \subseteq Y.\varphi$,

C2: $Y \subseteq Z$ implies $Y.\varphi \subseteq Z.\varphi$, and

C3: $Y.\varphi.\varphi = Y.\varphi$.

Read $Y.\varphi$ as "Y closure". A set Y is **closed** if $Y = Y.\varphi$. By C1, $\mathbf{U}$ itself must be closed. It is well known that these two definitions of closure are equivalent; the closed sets given by a closure operator on $\mathbf{U}$ form a closure system on $\mathbf{U}$ and, conversely, given a closure system $(\mathbf{U}, \mathcal{C})$, the closure operator on $\mathbf{U}$ is obtained by assigning to every subset $A \subseteq \mathbf{U}$ the intersection of all members of $\mathcal{C}$ that contain A as a subset.

Still a third characterization of closure systems can be found in [25]. We repeat their proof because it is not as widely known.

Proposition 1.1 *An expansive operator φ is a closure operator if and only if φ is path independent.*

Proof: Let φ be expansive and let $X, Y \subseteq \mathbf{U}$. Assume φ is a closure operator. Since $X, Y \subseteq X \cup Y$, $X.\varphi \cup Y.\varphi \subseteq (X \cup Y).\varphi$. And because $X \subseteq X.\varphi$ and $Y \subseteq Y.\varphi$, we have $(X.\varphi \cup Y.\varphi).\varphi \subseteq (X \cup Y).\varphi \subseteq (X.\varphi \cup Y.\varphi).\varphi$, so $(X \cup Y).\varphi = (X.\varphi \cup Y.\varphi).\varphi$.
Conversely, if φ is path independent (1), then $X \subseteq Y$ implies $X.\varphi \subseteq X.\varphi \cup Y.\varphi \subseteq (X.\varphi \cup Y.\varphi).\varphi = (X \cup Y).\varphi = Y.\varphi$. So φ is monotone. Finally, $Y.\varphi.\varphi = (Y.\varphi \cup Y.\varphi).\varphi = (Y \cup Y).\varphi = Y.\varphi$, so φ is idempotent. $\square$

Let $Y \subseteq \mathbf{U}$ be a closed set. A subset $X \subseteq Y$ is said to **generate** (or to be a **generating set** of) Y if $X.\varphi = Y$. By C3, every closed set Y generates itself. But, that tends to be uninteresting. We say a generating set X of Y is **non-trivial** if $X \subset Y$. In fact, we are really only interested in *minimal* non-trivial generating sets. More specifically, if X is a minimal generating set, it is called a **generator** of Y. A generator of Y will be denoted by $Y.\gamma$.[2] To define the generator operator γ on *all* subsets $Y \subseteq \mathbf{U}$, we let $Y.\gamma \equiv Y.\varphi.\gamma$. Readily, $Y.\gamma \subseteq Y$. When there is just a single minimal generator for any closed subset of $\mathbf{U}$, we say that $(\mathbf{U}, \varphi)$ is **uniquely generated**. A closure system $(\mathbf{U}, \varphi)$ is said to be **finitely generated** if for all closed subsets of $\mathbf{U}$, their generators are finite. Since for all $Y \subseteq \mathbf{U}$, $Y.\gamma \subseteq Y$, γ can also be regarded as a choice operator.

A closure operator/system can satisfy other axioms depending on the mathematical discipline. A *topological closure* is grounded and closed under finite union, *i.e.*, satisfying

C0: $\emptyset.\varphi = \emptyset$,

C4: $(Y \cup Z).\varphi = Y.\varphi \cup Z.\varphi$.

The closure operator of linear systems, often called the *spanning operator*, satisfies the Steinitz-MacLane *exchange axiom*

C5: if $p, q \notin Y.\varphi$, and $q \in (Y \cup \{p\}).\varphi$ then $p \in (Y \cup \{q\}).\varphi$.

Such closure systems are called *matroids*. Still other closure operators may satisfy an *anti-exchange axiom*

[2] When a closed set Y has more than one generator, the collection of all of them we denote by $Y.\Gamma = \{Y.\gamma_1, \ldots, Y.\gamma_n\}$ [15].

C6: if $p, q \notin Y.\varphi$, $p \neq q$, and $q \in (Y \cup \{p\}).\varphi$ then $p \notin (Y \cup \{q\}).\varphi$.

These closure operators, which include the geometric convex hull operator, are said to be **antimatroid closure** operators and the corresponding closure systems are called **antimatroids** or **convex geometries**. An important antimatroid property is the following.

Proposition 1.2 *Let $(\mathbf{U}, \varphi)$ be an antimatroid, let $Z \subseteq \mathbf{U}$ be a closed subset and let X, Y be generating sets of Z. If each of the sets X and Y contains a finite subset generating Z, then $X \cap Y$ is also a generating set of Z.*

Proof: We have $X.\varphi = Y.\varphi = Z$. Let $A \subseteq X$ and $B \subseteq Y$ be finite generating sets of Z and suppose that $X \cap Y$ does not generate Z, *i.e.*, that $(X \cap Y).\varphi \subset Z$. Put $C_0 = X \cap Y$. Then $A \not\subseteq C_0.\varphi$. Let $x_1 \in A - C_0.\varphi$ be an arbitrary point and put $C_1 = C_0 \cup \{x_1\}$. If $C_1.\varphi \subset Z$, then $A \not\subseteq C_1.\varphi$. Let $x_2 \in A - C_1.\varphi$ be an arbitrary point and put $C_2 = C_1 \cup \{x_2\}$. Repeating this construction, after a finite number k of steps we get a set $C_k = C_{k-1} \cup \{x_k\}$ such that $C_{k-1}.\varphi \subset Z$, $x_k \notin C_{k-1}.\varphi$ and $C_k.\varphi = Z$. Put $C_{k-1} = D_0$. Then $B \not\subseteq D_0.\varphi$. Let $y_1 \in B - D_0.\varphi$ be an arbitrary point and put $D_1 = D_0 \cup \{y_1\}$. If $D_1.\varphi \subset Z$, then $B \not\subseteq D_1.\varphi$. Let $y_2 \in B - D_1.\varphi$ be an arbitrary point and put $D_2 = D_1 \cup \{y_2\}$. Repeating this construction, after a finite number l of steps we get a set $D_l = D_{l-1} \cup \{y_l\}$ such that $D_{l-1}.\varphi \subset Z$, $y_l \notin D_{l-1}.\varphi$ and $D_l.\varphi = Z$. Thus, since $(D_{l-1} \cup \{y_l\}).\varphi = Z$, we have $x_k \in (D_{l-1} \cup \{y_l\}).\varphi$. But $(D_{l-1} \cup \{x_k\}).\varphi = Z$ holds too (because $(C_{k-1} \cup \{x_k\}).\varphi = Z$ and $C_{k-1} \subseteq D_{l-1}$), so we have $y_L \in (D_{l-1} \cup \{x_k\}).\varphi$. Suppose that $x_k \in D_{l-1}.\varphi$. Then $\{x_k\} \cup D_{l-1} \subseteq D_{l-1}.\varphi$, hence $(\{x_k\} \cup D_{l-1}).\varphi \subseteq D_{l-1}.\varphi$. It follows that $D_{l-1}.\varphi = Z$, which is a contradiction. Hence, $x_k \notin D_{l-1}.\varphi$. This contradicts the assumption that $(\mathbf{U}, \varphi)$ is an antimatroid. Therefore, $(X \cap Y).\varphi = Z$ and the proof is complete □

To see why the existence of finite generating subsets of X and Y is required in the previous statement, consider $\mathbf{U} = \mathbf{Z}$ (the set of integers) and let φ be the downset closure on the (naturally) ordered set $\mathbf{Z}$ (see section 1.5). Let X be the set of even integers and Y be the set of odd integers. Then both X and Y generate $\mathbf{U}$ but $X \cap Y$ does not because $(X \cap Y).\varphi = \emptyset.\varphi = \emptyset$.

Proposition 1.3 *Let $(\mathbf{U}, \varphi)$ be a finitely generated closure system. Then $(\mathbf{U}, \varphi)$ is an antimatroid if and only if $(\mathbf{U}, \varphi)$ is uniquely generated.*

Proof: If $(\mathbf{U}, \varphi)$ is not an antimatroid, then there exists some closed set $Y.\varphi$ with $p, q \notin Y.\varphi, p \neq q$, such that $p \in (Y \cup q).\varphi$ and $q \in (p \cup Y).\varphi$. Then p and q are members of distinct generators of $(Y \cup p).\varphi = (Y \cup q).\varphi$. The converse is a corollary of Prop. 1.2. □

Most closure systems occurring in applications are finite, and thus finitely generated..

If $(\mathbf{U}, \varphi)$ is a finite antimatroid, then the collection of all sets X_i with the same closure $Y = X_i.\varphi$ constitute a Boolean lattice with $Y.\gamma \subseteq X_i \subseteq Y$ [6, 25, 30]. In particular, if $X_i, \ldots X_k$ generate Y then $X_i \cap \ldots \cap X_k$ also generates Y by Proposition 1.1.

Closure systems are common in discrete applications, one encounters them in many guises. Three important types of closure operators are 1) convex hull operators, 2) poset closures, and 3) Galois closure of relations.

1.4 Convex Geometries

By a *convex geometry* we envision a collection of discrete points in an n-dimensional Euclidean space. For the illustrative purposes of this paper, we used a 2-space. The closure operator is the convex hull operator. A much more complete treatment can be found in [11, 12, 16]. There the closed sets are called *alignments* and their generators are called *extreme* points. The term "extreme points" is frequently use as a synonym for "generating set" in the literature.

A convex hull closure operator provides the quintessential example of antimatroid closure. Consider Figure 1. The point q is in the convex hull of $Y \cup \{p\}$, consequently p cannot

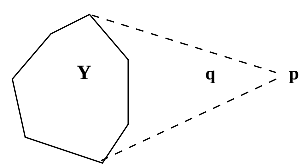

Figure 1: Illustration of antimatroid closure

possibly be in the convex hull of $Y \cup \{q\}$.

A closed convex hull is generated by elements of its boundary. The minimal generating set is precisely the vertices of the boundary, or its extreme points. Since, in a discrete geometry, these must be unique, convex hull closure is antimatroid. (Some authors treat any antimatroid closure space as a "convex geometry".) The efficiency of the Simplex method for solving linear inequalities [8] is based on this principle.

1.5 Posets as Closure Systems

Suppose $\mathbf{U}$ is a poset $(P, \leq)$ such as Figure 2, where $a < c < e$, *etc.* Or, equivalently, we may think of Figure 2 as an acyclic graph $G = (P, E)$. Two familiar closure operators on

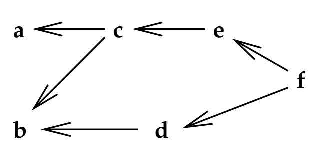

Figure 2: A 6 point poset

P are

$Y.\varphi_{\downarrow} = \{\, x \mid \exists y \in Y, x \leq y \,\}$ and

$Y.\varphi_{\uparrow} = \{\, z \mid \exists y \in Y, y \leq z \,\}$.

"Downset closure", $\varphi_{\downarrow}$, on Figure 2 yields precisely the closure system $\mathcal{C}_U$ introduced in Section 1.3 as (2).[3] $Y.\varphi_{\downarrow}$ is sometimes called an ideal, and $Y.\varphi_{\uparrow}$ a filter.

A potentially more interesting closure operator on posets is the **convex**, or interval, **closure** operator, denoted by φ_C and defined by $Y.\varphi_C = \{\, x \mid \exists y_1, y_2 \in Y, y_1 \leq x \leq y_2\}$. Some authors call these closed sets, which play a role in quantum theory, "causal sets" [5, 36, 40]. Because, for all $Y \subseteq \mathbf{U}$, $Y.\varphi_C \subseteq Y.\varphi_{\downarrow}$ we sometimes call φ_C a **finer** closure operator. It is easy to show that $\varphi_C = \varphi_{\downarrow} \cap \varphi_{\uparrow}$. We call $\varphi_{\downarrow}, \varphi_{\uparrow}$, and φ_C collectively **path closures**. All of the path closures $\varphi_{\downarrow}, \varphi_{\uparrow}$ and φ_C are antimatroid closures, and $\varphi_{\downarrow}$ and $\varphi_{\uparrow}$ are, in addition, topological closures.

It is common to partially order the closed sets of any closure operator φ with respect to inclusion, thus creating a lattice $\mathcal{L}_{\varphi}$. If φ is antimatroid, $\mathcal{L}_{\varphi}$ is lower semi-modular [23] or meet distributive [11]. We can also partially order *all* the subsets of $\mathbf{U}$ with respect to the closure φ, by

$$X \sqsubseteq Z \quad \text{if and only if} \quad Z \cap X.\varphi \subseteq X \subseteq Z.\varphi \tag{3}$$

for all $X, Z \subseteq \mathbf{U}$. It is not hard to show that $\sqsubseteq$ really is an ordering; only transitivity takes a bit of thought [30]. The following proposition can be found in [25, 30].

Proposition 1.4 *Let $(\mathbf{U}, \varphi)$ be an antimatroid and $Y, Z \subseteq \mathbf{U}$ subsets, Z closed. Then Z covers Y with respect to $\sqsubseteq$ if and only if $Y = Z - p$ for some $p \in Z.\gamma$.*

By Proposition 1.4, deletion of any generating (extreme) point, say f, from $Z = abcdef$ in Figure 2 must create a new closed set $Y = abcde$. In [19], this repeated deletion of generating/extreme points is called "shelling" where they used it to define antimatroid behavior.

1.5.1 Neighborhood Closure

Social relationships are often represented by undirected graphs, $(\mathbf{U}, \mathcal{A})$, such as Figure 3, where $\mathcal{A}$ is a reflexive, symmetric adjacency relation. Here, the **neighborhood operator**, η is defined by, $Y.\eta = \{x | (x, y) \in \mathcal{A}, y \in Y\}$. By the **neighborhood closure operator**, φ_{η}, we mean

$$Y.\varphi_{\eta} = \{z | z.\eta \subseteq Y.\eta\} \tag{4}$$

[3] If we draw partial orders from left to right, as in Figure 2, perhaps this should be called a "left set" ordering!

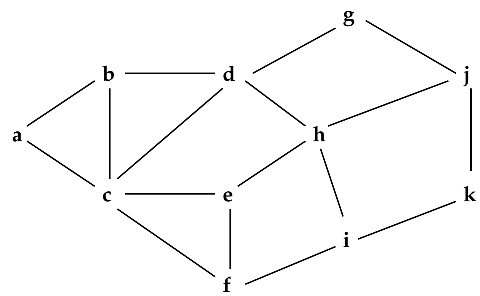

Figure 3: An undirected graph.

Readily, for all Y, $Y \subseteq Y.\varphi_\eta \subseteq Y.\eta$, and $\mathbf{U}.\varphi_\eta = \mathbf{U}$. In Figure 3, the set $Y = cgi$ generates the entire closed system $(\mathbf{U}, \varphi_\eta)$. It is not difficult to show that φ_η really is a closure operator [33]. This neighborhood closure operator can be iterated to approximate a discrete convex hull, as in [37], where it was used to "clean" discrete binary data; and it can be used to reduce a network to its fundamental, non-chordal cycles [32].

Neighborhood closure need not be antimatroid.

1.5.2 Galois Closure

Galois closure has a long history [13, 26]; but has now entered the main stream of applied mathematics with [9, 43]. Of particular interest to the social sciences has been the development of formal concept analysis as introduced by Wille [14, 44]. "Concepts" constitute natural clusters of individuals and their attributes. Partially ordered by containment, these closed concept clusters form a "concept lattice" which has been used in a variety of applications ranging from data mining [28, 42] to analyzing legacy codes [21] to extracting rules for AI systems [31].

Galois closure need not be antimatroid.

1.5.3 Koshevoy Closure

Gleb Koshevoy [20] established a fundamental duality between choice and closure operators. The generators γ (extreme points) of an antimatroid closure φ constitute a path independent choice function σ on $\mathbf{U}$, and conversely. More precisely he showed that

Proposition 1.5 *If φ is an antimatroid closure operator, then for all $X, Y \subseteq \mathbf{U}$,*

$$(X \cup Y).\gamma = (X.\gamma \cup Y.\gamma).\gamma.$$

(Recall that for all Y, $Y.\gamma = Y.\varphi.\gamma$.) By Proposition 1.1, every closure operator φ must itself be path independent, but the generating operator γ will have this property only if φ

is antimatroid.

To establish this relationship Koshevoy created the operator κ_α which is now called the Koshevoy operator. Specifically, given any operator α on $\mathbf{U}$ then

$$Y.\kappa_\alpha = \bigcup \{X \subseteq \mathbf{U}, X.\alpha = Y.\alpha\} \tag{5}$$

Readily κ_α is a kind of inverse operator, similar to what we will consider in Section 2.2, and if the operator α is path independent then the union on the right side of (5) is well behaved and $Y.\kappa_\alpha.\alpha = Y.\alpha$. Because κ_α is expansive, we have from [25] that

Proposition 1.6 *κ_α is a closure operator, that is monotone and idempotent, if and only if α is path independent.*

Consequently, in light of Proposition 1.1, κ_α is path independent if and only if α is path independent. More importantly, we have [6, 20]

Proposition 1.7 *σ is a path independent choice operator if and only if for all Y $Y.\sigma = Y.\gamma$ where γ is the generating operator of an antimatroid closure.*

In summary, provided $\mathbf{U}$ is finite or finitely generated, all closure operators φ are path independent (Prop. 1.1); φ is antimatroid if and only if γ is path independent (Prop. 1.7); or equivalently γ is a single valued operator (Prop. 1.3).

If $\mathcal{S} = (\mathbf{U}, \alpha)$ is any system in which α is path independent, we may use the Koshevoy operator κ_α to induce a closure structure on $\mathcal{S}$. Associative binary operators are path independent. Since we will define transformations in the next section in terms of closure, the Koshevoy operator extends the relevance of that development to many more discrete systems.

The duality between antimatroid closure and path independent choice operators, expressed in Proposition 1.7, has been well explored [6, 20, 25]. If the choice operator has the additional properties of "concordance" ($X.\sigma \cap Y.\sigma \subseteq (X \cup Y).\sigma$) and "heritage" ($X \subseteq Y$ implies $Y.\sigma \cap X \subseteq X.\sigma$) it is said to be "rationalizable" [2, 24].[4] A rationalizable choice operator $Y.\sigma$ can be represented as the maximal elements of Y in a partial order. That is, the dual can be the generators of a path closure such as $\varphi_\downarrow$ on Figure 2.

With an abundance of closure operators[5] whether the duals of choice operators, whether induced by the Koshevoy operator, or simply defined in their own right, it seems natural to let the closed sets of discrete spaces assume much the same role as open sets in continuous domains. In the next section we define continuous and closed transformations of discrete spaces with respect to the behavior of the closed sets.

[4] Choice operators may also have the "outcast" property ($X \subseteq Y, Y.\sigma \subseteq X$ implies $X.\sigma = Y.\sigma$). "Concordance", "heritage" and "outcast" are independent properties [2, 24].

[5] On a set $\mathbf{U}$ of $n > 10$ elements, there are many more than n^n distinct antimatroid closure systems.

2 Transformations on Discrete Systems

A transformation f of a closure system $\mathcal{C}$ into another $\mathcal{C}'$ is a function taking the power set of $\mathbf{U}$ into that of $\mathbf{U}'$. To simplify notation, we denote these functions by $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ instead of the more correct $2^{\mathbf{U}} \xrightarrow{f} 2^{\mathbf{U}'}$ and we call these functions "transformations" to emphasize their "set valued" nature. If $\mathbf{U}' = \mathbf{U}$ then the transformation f is usually called an operator, α. Closure, φ, and choice, σ, can thus be regarded as transformations. To emphasize the distinction, we use the greek alphabet to denote operators and the latin alphabet to denote transformations. A focus of this paper is the behavior of operators under transformations.

A transformation $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ is said to be **monotone** if $\forall X, Y \subseteq \mathbf{U}$, $X \subseteq Y$ implies $X.f \subseteq Y.f$.[6]

It is evident that the identity transformation is monotone and that the composition of monotone transformations is monotone.

2.1 Continuous and Closed Transformations

A transformation f simply maps subsets of $\mathbf{U}$ into subsets of $\mathbf{U}'$. We ask "what happens to the closure properties of $\mathbf{U}$ under f". To emphasize this, we will usually denote a transformation by $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$. The question we address in this section is "can we characterize those transformations which preserve the closure structure?", that is for which $Y.\varphi.f = Y.f\varphi'$.

The most commonly accepted definition of "continuity" in discrete closure spaces is that a transformation $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ is **continuous** if $\forall Y \subseteq \mathbf{U}$, $Y.\varphi.f \subseteq Y.f.\varphi'$ [27, 43].

We will say a transformation $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ is **closed** if f takes closed sets in $(\mathbf{U}, \varphi)$ onto closed sets in $(\mathbf{U}', \varphi')$.

Clearly, the identity transformation of a closure system is continuous and closed.

The transformation properties of monotonicity, continuity and closedness are independent, as shown by the following Figure 4. Here $\mathbf{U} = \{a, b\}$, $\mathbf{U}' = \{x, y, z\}$, $\mathcal{C}_\varphi = \{ab\}$, $\mathcal{C}_{\varphi'} = \{xy, xyz\}$. In the first three cases each f_i satisfies one property, but not the other two. The f_4 transformation is monotone and closed (as can be easily verified), but it is not continuous.

It is obvious that the composition of closed transformations is a closed transformation. On the other hand, the same is not true for continuous transformations as the following example shows:

Let $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$, $(\mathbf{U}', \varphi') \xrightarrow{g} (\mathbf{U}'', \varphi'')$ be the transformations where $\mathbf{U} = \{a, b\}$, $\mathcal{C}_{\mathbf{U}} = \{\emptyset, ab\}$; $\mathbf{U}' = \{x, y, z\}$, $\mathcal{C}_{\mathbf{U}'} = \{\emptyset, y, z, yz, xyz\}$; and $\mathbf{U}'' = \{p, q\}$, $\mathcal{C}_{\mathbf{U}''} = \{\emptyset, p, pq\}$.

[6] Since operators are transformations, this is identical to the definition of monotonicity in Section 1.1.

$Y \in \mathbf{U}$	$Y.f_1$	$Y.f_2$	$Y.f_3$	$Y.f_4$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
a	xy	y	xy	x
b	xyz	y	xyz	y
ab	x	yz	xy	xyz
	$Cont$	$\neg Cont$	$\neg Cont$	$\neg Cont$
	$\neg Mono$	$Mono$	$\neg Mono$	$Mono$
	$\neg Clos$	$\neg Clos$	$Clos$	$Clos$

Figure 4: Four transformations, $\mathbf{U} \xrightarrow{f_k} \mathbf{U}'$

We define f by $\emptyset.f = \emptyset$, $\{a\}.f = \{x\}$, $\{b\}.f = \mathbf{U}.f = \{x, y\}$ and g by $\emptyset.g = \emptyset$, $\{x.y\}.g = \mathbf{U}''$ and $A.g = \{p\}$ whenever $A \subseteq \mathbf{U}'$ is a subset with $\emptyset \neq A \neq \{x, y\}$. Then both f and g are continuous but $f \cdot g$ is not because $\{a\}.\varphi.(f \cdot g) = \mathbf{U}''$ while $\{a\}.(f \cdot g).\varphi'' = \{p\}$.

But we have

Proposition 2.1 *Let* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$, $(\mathbf{U}', \varphi') \xrightarrow{g} (\mathbf{U}'', \varphi'')$ *be transformations and let* g *be monotone. If both* f *and* g *are continuous, then so is* $\mathbf{U} \xrightarrow{f \cdot g} \mathbf{U}''$.

Proof: We have $X.\varphi.f \subseteq X.f.\varphi'$ for any $X \in \mathbf{U}$ and $Y.\varphi'.g \subseteq Y.g.\varphi''$ for any $Y \in \mathbf{U}'$. Consequently, as g is monotone, $X.\varphi.f.g \subseteq X.f.\varphi'.g \subseteq X.f.g.\varphi''$. This means that $f \cdot g$ is continuous. $\square$

As illustrated by Proposition 2.1, monotonicity is often required for a transformation, or operator, to be well behaved. In the remainder of this paper we will consider only monotone transformations.

Proposition 2.2 *A monotone transformation* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is closed if and only if* $\forall X \subseteq \mathbf{U},\ X.f.\varphi' \subseteq X.\varphi.f$.

Proof: Let f be closed. By monotonicity, $X \subseteq X.\varphi$ implies $X.f \subseteq X.\varphi.f$. But, because $X.\varphi$ is closed and f is closed, $X.f.\varphi' \subseteq X.\varphi.f$
Conversely, let all subsets $X \subseteq \mathbf{U}$ fulfill $X.f.\varphi' \subseteq X.\varphi.f$ and let X be a closed subset of $(\mathbf{U}, \varphi)$. Then $X.f.\varphi' \subseteq X.f$. But, readily $X.f \subseteq X.f.\varphi'$, so equality holds. $\square$

Theorem 2.3 *A monotone transformation* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is closed and continuous if and only if, for all* $X \subseteq \mathbf{U}$, $X.\varphi.f = X.f.\varphi'$.

Proof: Proposition 2.2 and the definition of continuity $\square$

Theorem 2.3 provides necessary and sufficient conditions on a monotone transformation f to ensure that f commutes with the closure operator φ as in Figure 5.

$$\begin{tikzcd} \mathbf{U} \arrow[r, "f"] \arrow[d, "\varphi"] & \mathbf{U}' \arrow[d, "\varphi'"] \\ (\mathbf{U}, \varphi) \arrow[r, "f"] & (\mathbf{U}', \varphi') \end{tikzcd}$$

Figure 5: f regarded as a closure system transformation

2.2 Transformation Inverse Properties

An important property of "continuity" is that the inverse image of closed sets ought to be closed. Unfortunately, this is not a property of continuity by itself. Consider f_1 in Figure 4 which is continuous. The set $\{xy\}$ is closed in $\mathbf{U}'$, but $\{xy\}.f_1^{-1} = \{a\}$ which is not closed. However, for monotone continuous transformations we have

Proposition 2.4 *Let* $(P, \varphi) \xrightarrow{f} (P', \varphi')$ *be monotone, continuous and let* $Y' = Y.f$ *be closed. Then* $Y.\varphi.f = Y'$.

Proof: Let $Y.f$ be closed in P'. Because f is continuous $Y.\varphi.f \subseteq Y.f.\varphi' = Y.f$, since $Y.f$ is closed. By monotonicity, $Y.f \subseteq Y.\varphi.f$, so $Y.\varphi.f = Y.f$. $\Box$

The proposition does not assert that for any closed set $Y' \subseteq \mathbf{U}'$ that $Y'.f^{-1}$ will exist, even if $\mathbf{U}.f = \mathbf{U}'$, but if the pre-image of a closed set does exist, it must also be, in a sense, closed since $Y.\varphi.f = Y.f$.

A common way of defining a transformation $(\mathbf{U}, E) \xrightarrow{f} (\mathbf{U}', E')$ is to first define $\{y\}.f$ for all singleton sets in $\mathbf{U}$, *i.e.* as a function on $\mathbf{U}$, and then extend this to all $Y \subseteq \mathbf{U}$ by $Y.f^+ = \bigcup_{y \in Y} \{y\}.f$.[7] We call f^+ an **extended transformation**, *provided* $\mathbf{U}.f^+ = \mathbf{U}'$. Any extended transformation is by construction, monotonic.

Proposition 2.5 *If* $(\mathbf{U}, E) \xrightarrow{f^+} (\mathbf{U}', E')$ *is an extended transformation, then for all* $y' \in Y' = Y.f^+$ *there exists* $y \in Y$ *such that* $y' \in \{y\}.f^+$.

Proof: Let $y' \in Y'$. By the extended construction $Y' = \bigcup_{y \in Y} \{y\}.f$, hence $y' \in \{y\}.f^+$ for some $y \in Y$. $\Box$

Note that this is quite different from asserting a true inverse existence, that for all $y' \in Y'$, there exists some $y \in Y$ such that $y.f = y'$. To get some sense of the import of this "weak inverse existence" proposition, consider the simple transformation f of Figure 6. If we define

[7] f^+ is sometimes said to be "lifted" from f [38].

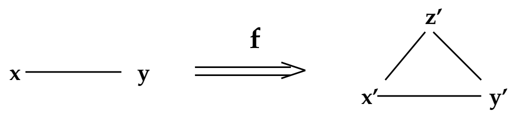

Figure 6: A simple transformation f with multiple definitions.

f on $\mathbf{U}$ by $x.f = x'$ and $y.f = y'$, then by extension $\{xy\}.f^+ = x'y'$ and z' has no pre-image; so $\mathbf{U}.f \neq \mathbf{U}'$. However, if we let $x.f = \{x'z'\}, y.f = \{y'z'\}$ then $\{xy\}.f^+ = x'y'z'$. Now $\mathbf{U}.f = \mathbf{U}'$, so f^+ is an extended transformation, and Proposition 2.5 is clearly satisfied. We now have $z' \in x.f$ as well as $z' \in y.f$, but $z'.f^{-1} \neq \{xy\}$.

2.3 Homomorphisms of Posets

Recall that $f : P \to P'$ is a **homomorphism** of the poset P into P' if $x \leq y$ in P implies $f(x) \leq f(y)$ in P'.[8] Since an anti-chain can be the pre-image of any ordered set under a homomorphism, it would be an exaggeration to call them "structure preserving". Often one wants the structure of the domain to have some resemblance to its image. For this reason some authors prefer "strong homomorphisms" where a homomorphism is called *strong* if in addition $x' \leq y'$ in P' implies $\exists x, y \in P$ such that $f(x) = x', f(y) = y'$ and $x \leq y$.

Proposition 2.6 *A function $f : (P, \leq) \to (P', \leq')$ is a homomorphism if and only if the extended transformation $(P, \varphi) \xrightarrow{f^+} (P', \varphi')$ is continuous with respect to a path closure.*

Proof: Let f be a homomorphism and let $Y \subseteq P$. Let $x' \in Y.\varphi_{\downarrow}.f^+$. Then $\exists x, y, \in Y.\varphi_{\downarrow}$ where $x \leq y$ and $x' = f(x)$. Consequently, $f(x) \leq f(y)$. Therefore, $x' = f(x) \in Y.f^+.\varphi_{\downarrow}$. So $Y.\varphi_{\downarrow}.f^+ \subseteq Y.f^+.\varphi_{\downarrow}$.
Conversely, assume that f^+ is continuous and $x \leq y$. Then $x \in y.\varphi_{\downarrow}$ and $x.\varphi_{\downarrow} \subseteq y.\varphi_{\downarrow}$. $\{x\}.f \subseteq \{x\}.\varphi_{\downarrow}.f^+ \subseteq \{y\}.\varphi\downarrow.f^+ \subseteq \{y\}.f^+.\varphi_{\downarrow}$. Thus $f(x) \leq f(y)$.
The proof when the closure is φ_C or $\varphi_{\uparrow}$ is similar. □

In an early application of this principle [29] it was shown that if $f : G \to G'$ was any poset homomorphism, then G' would be acyclic if and only if the inverse image of every point in G' was a convex (in the sense of φ_C of Section 1.5) set in G.

Poset homomorphisms need not be closed. In Figure 7, $\{a, b, d\}$ is closed (assuming $\varphi_{\downarrow}$ closure), but $\{a', b', d'\}$ is not. Even strong homomorphisms need not be closed, as shown by Figure 8, because $\{b_2\}$ is closed but $\{b_2\}.g = \{b'\}$ is not. However, we can show

[8] Note that we use the traditional prefix notation $f : \mathbf{U} \to \mathbf{U}'$ for point functions on $\mathbf{U}$, and $\mathbf{U} \xrightarrow{f^+} \mathbf{U}'$ for extended transformations between the power sets.

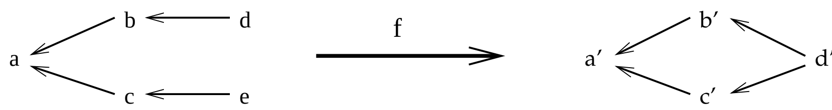

Figure 7: A homomorphism f that is not closed.

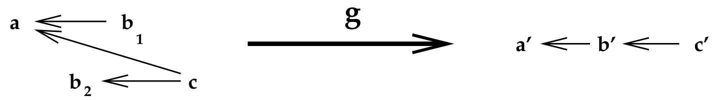

Figure 8: A strong homomorphism g that is not closed.

Proposition 2.7 *Let $g : (P, \leq) \to (P', \leq)$ be a surjective homomorphism. If $(P, \varphi) \xrightarrow{g^+} (P, \varphi')$ is closed with respect to a path closure, then g is a strong homomorphism.*

Proof: Let g^+ be closed, let $x' \leq y'$ in P' and let $\{y'\}.g^{-1} = Y$. Then $Y.g^+ = \{y'\}$, so $x' \in Y.g^+.\varphi_{\downarrow} \subseteq Y.\varphi_{\downarrow}.g^+$ implying there exists $y_k \in Y$ and $x_i \in Y.\varphi_{\downarrow}$ such that $x_i \leq y_k$ where $x_i.g = x'$ and by definition of Y, $y_k.g = y'$. So g^+ is strong. $\Box$

Combining Prop. 2.6 with Prop. 2.7 we readily have that if g^+ is closed, then g is a homomorphism so g^+ must also be continuous. But, only in this special case where we have extended transformations over poset closures need this be true. It is not a general implication.

2.4 Intersection of Transformations

Given a family f_i, $i \in I$, of numeric functions (with the same domain and range), one can define the operators $\sum_{i \in I} f_i$ of sum and $\prod_{i \in I} f_i$ of product of this family in a pointwise fashion; that is $\sum_{i \in I} f_i$ and $\prod_{i \in I} f_i$ are defined $\forall x$ to be the sum $\sum_{i \in I} f_i(x)$ and the product $\prod_{i \in I} f_i(x)$, respectively. Since domains of our transformations are power sets, we may define the intersection, $\bigcap_{i \in I} f_i$, or meet, of a family $\mathbf{U} \xrightarrow{f_i} \mathbf{U}'$, $i \in I$, of transformations in a "set-wise" fashion, *i.e.*, as $Y.\bigcap_{i \in I} f_i = \bigcap_{i \in I} Y.f_i$ for each $Y \subseteq \mathbf{U}$. Earlier we had observed that $\varphi_C = \varphi_{\downarrow} \cap \varphi_{\uparrow}$.

Proposition 2.8 *If $(\mathbf{U}, \varphi) \xrightarrow{f_i} (\mathbf{U}', \varphi')$, $i \in I$ is a family of monotone transformations, then $\bigcap_{i \in I} f_i$ is monotone.*

Proof: Let f_i, $i \in I$, be monotone and let $X, Y \subseteq \mathbf{U}$ be a subsets. Then $Y.\bigcap_{i \in I} f_i = \bigcap_{i \in I} Y.f_i \subseteq \bigcap_{i \in I} Z.f_i = Z.\bigcap_{i \in I} f_i)$. $\Box$

Proposition 2.9 *If* $(\mathbf{U},\varphi) \xrightarrow{f_i} (\mathbf{U}',\varphi')$, $i \in I$ *is a family of closed transformations, then* $\bigcap_{i\in I} f_i$ *is closed.*

Proof: Let f_i, $i \in I$, be closed and let $Y \subseteq \mathbf{U}$ be a subset. Then $Y.\varphi.\bigcap_{i\in I} f_i = \bigcap_{i\in I} Y.\varphi.f_i \supseteq \bigcap_{i\in I} Y.f_i.\varphi' \supseteq (\bigcap_{i\in I} Y.f_i).\varphi' = Y.(\bigcap_{i\in I} f_i).\varphi'$. □

However, continuity is not closed under (*i.e.*, preserved by) intersections as the following example shows:

Let $\mathbf{U} = \{a,b\}$ and $\mathbf{U}' = \{x,y,z\}$. Let $\{a\}.\varphi = \mathbf{U}$ and $\{b\}.\varphi = \{b\}$. Let, $\{x\}.\varphi' = \{x,z\}$, $\{y\}.\varphi' = \{y,z\}$, $\{x,y\}.\varphi' = \mathbf{U}$ and $X.\varphi = X$ for all the other subsets $X \subseteq \mathbf{U}$. Let $(\mathbf{U},\varphi) \xrightarrow{f,g} (\mathbf{U}',\varphi')$ be the transformations given by $\{a\}.f = \{x\}$, $\{b\}.f = \{z\}$, $\emptyset.f = \emptyset$, $\mathbf{U}.f = \{x,z\}$, and $\{a\}.g = \{y\}$, $\{b\}.g = \{z\}$, $\emptyset.g = \emptyset$, $\mathbf{U}.g = \{y,z\}$. Then both f and g are continuous, $\{a\}.\varphi.(f \cap g) = \{z\}$ but $\{a\}.(f \cap g).\varphi' = \emptyset$. So, $f \cap g$ is not continuous. Even requiring the transformations to be monotone and continuous does not help because both f and g are monotone.

The transformations from $\mathbf{U}$ into $\mathbf{U}'$ may be identified with the disjoint unions $\coprod_{Y\subseteq U} Z'_Y = \bigcup_{Y\subseteq U}(\{Y\}\times Z'_Y)$ where $Z'_Y \subseteq \mathbf{U}'$ for all $Y \subseteq \mathbf{U}$. Namely, every such a disjoint union gives the transformation $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ with $Y.f = Z'_Y$ for all $Y \subseteq U$ and conversely. Given a pair f,g of transformations from $\mathbf{U}$ into $\mathbf{U}'$, we clearly have $f \subseteq g$ if and only if $Y.f \subseteq Y.g$ for each subset $Y \subseteq \mathbf{U}$. The smallest (with respect to the set inclusion) of all transformations from $\mathbf{U}$ into $\mathbf{U}'$ is the transformation $\mathbf{U} \xrightarrow{\mathcal{O}} \mathbf{U}'$ given by $Y.\mathcal{O} = \emptyset$ for each $Y \subseteq \mathbf{U}$, *i.e.*, $\mathcal{O} = \coprod_{Y\subseteq U} Y.f$ where $Y.f = \emptyset$ for each $Y \subseteq \mathbf{U}$. Thus, we have $\mathcal{O} = \emptyset$. On the other hand, the greatest of all transformations from $\mathbf{U}$ into $\mathbf{U}'$ is the transformation $\mathbf{U} \xrightarrow{\mathcal{I}} \mathbf{U}'$ given by $Y.\mathcal{I} = \mathbf{U}'$ for each $Y \subseteq \mathbf{U}$, *i.e.*, $\mathcal{I} = \coprod_{Y\subseteq U} Y.f$ where $Y.f = \mathbf{U}'$ for each $Y \subseteq \mathbf{U}$. Of course, both $\mathcal{O}$ and $\mathcal{I}$ are monotone transformations. Note that the points of $\mathcal{I}$ coincide with the transformations $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ for which there exists a subset $Y_0 \subseteq \mathbf{U}$ such that $Y_0.f$ is a singleton and $Y.f = \emptyset$ for all $Y \subseteq U$, $Y \neq Y_0$. Obviously, each transformation from $\mathbf{U}$ to $\mathbf{U}'$ may be obtained as the disjoint union of some of these "atomic" transformations.

Given a pair $\mathcal{C}_\infty$, $\mathcal{C}_\in$ of closure systems, we denote by $[\mathcal{C}_1,\mathcal{C}_2]$ the set of all closed transformations from $\mathcal{C}_1$ into $\mathcal{C}_2$. Clearly, the transformation $\mathcal{I}$ (from $\mathcal{C}_1$ into $\mathcal{C}_2$) belongs to $[\mathcal{C}_1,\mathcal{C}_2]$. Now, Proposition 2.9 results in

Proposition 2.10 *Let* $\mathcal{C}_1,\mathcal{C}_2$ *be closure systems. Then* $[\mathcal{C}_1,\mathcal{C}_2]$ *is a closure system on the universe* $\mathcal{C}_\infty \xrightarrow{\mathcal{I}} \mathcal{C}_\in$.

Transformations of discrete closure systems have so many intriguing properties that it is natural to ask about their categorical properties. This we do in the next section.

3 Categories of Closure Systems with Transformations as Morphisms

For the categorical terminology used see [1] and [34].

The category of grounded closure systems and continuous (point) functions has been studied by many authors, see e.g. [7] and [10]. Since this category is not cartesian closed, in [7] the authors construct its cartesian closed extension by adding new objects and morphisms between them. In our approach, instead of point functions, transformations are used as morphisms between closure systems. As described in Section 2, all point functions can be simply lifted to become transformations, so that transformations can be considered to be extensions of point functions. Using this kind of extension, we will find a cartesian closed category whose objects are just the closure systems.

All categories considered in this section will have all closure systems as the class of objects and certain transformations as morphisms. Thus, these categories may be given by just specifying the transformations that are their morphisms. By results of Section 2, all monotone continuous transformations constitute a category - we denote it by **MCont** - and also all monotone closed transformations constitute a category - we denote it by **MClo**. Clearly, both **MCont** and **MClo** are concrete categories over **Set** with the underlying functor F given by $F(\mathbf{U}, \varphi) = 2^{\mathbf{U}}$ for objects and $Ff = f$ for morphisms. The closure system $\mathcal{C}_0$ with the empty universe is the unique **terminal object** in both **MCont** and **MClo** because the transformation f defined by $X.f = \emptyset, \forall X \in \mathcal{C}$ is a unique monotone, continuous and closed transformation from $\mathcal{C}$ into $\mathcal{C}_0$. Observe that in the category **Set**, the empty set is the initial object and singletons are terminal objects [34]. The difference is that, in the category of closure systems, morphisms act on power sets of universes and not on the universes themselves.

Let $\mathcal{C}_i$, $i \in I$ (I a set), be a family of closure systems on the base sets $\mathbf{U}_i$. We let the disjoint union $\coprod_{i\in I} \mathbf{U}_i$ be the base set of the **direct product** $\prod_{i\in I} \mathcal{C}_i$ whose closed sets are defined to be just the sets of the form $\coprod_{i\in I} X_i$ where $X_i \in \mathcal{C}_i$ are closed sets for each $i \in I$. Thus, if $\prod_{i\in I} \mathcal{C}_i = (\coprod_{i\in I} \mathbf{U}_i, \varphi)$ and $\mathcal{C}_i = (\mathbf{U}_i, \varphi_i)$, we have $\varphi_i = \varphi|_{U_i}$ (for each $i \in I$). If $I = \{i_1, i_2\}$, we write $X_{i_1} \uplus X_{i_2}$ and $\mathcal{C}_{i_1} \times \mathcal{C}_{i_2}$ instead of $\coprod_{i\in I} X_i$ and $\prod_{i\in I} \mathcal{C}_i$, respectively.

Proposition 3.1 *The direct product is a concrete product in both* **MCont** *and* **MClo**.

Proof: Let $\mathcal{C}_i$, $i \in I$, be a family of closure systems. Of course, if $I = \emptyset$, then $\prod_{i\in I} \mathcal{C}_i$ is the terminal object. So, we may suppose that $I \neq \emptyset$. First, we will show that $\prod_{i\in I} \mathcal{C}_i$ is a closure system. Let Z_j, $j \in J$, be closed sets in $\prod_{i\in I} \mathcal{C}_i$. Then, for each $j \in J$, $Z_j = \coprod_{i\in I} X_i^{(j)}$ for some $X_i^{(j)} \in \mathbf{U}_i$, $i \in I$. We have $\bigcap_{j\in J} Z_j = \bigcap_{j\in j} \coprod_{i\in I} X_i^{(j)} = \coprod_{i\in I} \bigcap_{j\in J} X_i^{(j)}$, hence $\bigcap_{j\in J} Z_j$ is closed in $\prod_{i\in I} \mathcal{C}_i$. We can now define **projection** morphisms $\prod_{i\in I} \mathcal{C}_i \xrightarrow{\pi_i} \mathcal{C}_i$, $i \in I$, by $Z.\pi_i = Z \cap \mathbf{U}_i$. Clearly, π_i is a monotone transformation for each $i \in I$. But π_i is also continuous for each $i \in I$ because, given a subset $Z \subseteq \coprod_{i\in I} U_i$, we have $Z.\varphi.\pi_i = Z.\varphi \cap U_i = Z.\pi_i.\varphi_i$.

Now, let $\mathcal{C} = (\mathbf{U}, \psi)$ be a closure system and let $\mathcal{C} \xrightarrow{f_i} \mathcal{C}_i$, $i \in I$, be morphism in **MCont**. Let $\mathcal{C} \xrightarrow{h} \prod_{i\in I} \mathcal{C}_i$ be the transformation given by $\forall Z \in \mathcal{C} :\ Z.h = \coprod_{i\in I} Z.f_i \in \prod_{i\in I} \mathcal{C}_i$. Clearly, $f_i = h \cdot \pi_i$ for each $i \in I$ and h is a unique transformation having this property. It is obvious that h is monotone (because f_i, $i \in I$, are monotone). For every subset $Z \subseteq \mathcal{C}$ we have $Z.\psi.h = \coprod_{i\in I} Z.\psi.f_i \subseteq \coprod_{i\in I} Z.f_i\varphi_i \subseteq (\coprod_{i\in I} Z.f_i).\varphi_i = Z.h.\varphi_i$. Therefore, h is continuous. We have shown that $\prod_{i\in I} \mathcal{C}_i$ is a product in **MCont**.
Obviously, given a family $\mathcal{C}_i$, $i \in I$ of closure systems, the projections $\prod_{i\in I} \mathcal{C}_i \xrightarrow{\pi_i} \mathcal{C}_i$ are closed for all $i \in I$. Further, let $\mathcal{C} = (\mathbf{U}, \psi)$ be a closure system and $\mathcal{C} \xrightarrow{f_i} \mathcal{C}_i$, $i \in I$, be closed morphisms. Let $Z \subseteq \mathcal{C}$ be a closed subset. Then $Z.h = \coprod_{i\in I} Z.f_i$ and, since f_i $(i \in I)$ are closed, $Z.f_i$ $(i \in I)$ are closed too. Consequently, $Z.h$ is closed. Thus, with respect to the above considerations, $\prod_{i\in I} \mathcal{C}_i$ is a product in **MClo**.
It is clear that $\prod_{i\in I} \mathcal{C}_i$ is a concrete product in each of the categories **MCont** and **MClo** because $\mathrm{F}(\prod_{i\in I} \mathcal{C}_i) = 2^{\coprod_{i\in I} \mathcal{C}_i} \cong \prod_{i\in I} 2^{\mathcal{C}_i} = \prod_{i\in I} F(\mathcal{C}_i)$. $\square$

Since the direct product $\prod_{i\in I} \mathcal{C}_i$ in **MCont** and **MClo** is a concrete product over **Set**, it is clear now why its universe is a coproduct of the universes of the individual closure systems $\mathcal{C}_i$, $i \in I$.

Of course, it immediately follows from Proposition 3.1 that $\prod_{i\in I} \mathcal{C}_i$ is a concrete product also in **MCont** $\cap$ **MClo**, *i.e.*, in the category of closure systems with monotone continuous and closed transformations as morphisms (see Theorem 2.3). While this category is a subcategory of **MClo**, there is a supercategory **Clo** of **MClo** obtained by omitting the requirement of monotonicity of the morphisms. Thus, **Clo** has just the closed transformations as morphisms. It is clear from the proof of Proposition 3.1 that the direct product $\prod_{i\in I} \mathcal{C}_i$ is a concrete product also in **Clo**. The category **Clo** is especially interesting because we will show now that it is a so-called cartesian closed category.

Recall that a category **C** is **cartesian closed** [18, 34] if it has finite products and for any two objects $A, B \in \mathbf{C}$ there exists an object B^A in **C** and a morphism $B^A \times A \xrightarrow{ev} B$ with the property that for each $C \times A \xrightarrow{f} B$ there exists a unique morphism $C \xrightarrow{g} B^A$ such that $(g \times id_a) \cdot ev = f$, *i.e.*, such that the diagram on Fig. 12 commutes. The morphism ev is then said to be the evaluation morphism and the object B^A is called the *exponential object*.

Thus, in addition to finite multiplication, cartesian closed categories have also the operation of exponentiation on objects and these two operations are well behaved, *i.e.*, they fulfill analogies of the usual laws which are valid for multiplication and exponentiation of natural numbers (the most important of them is the so-called first exponential law $(A^B)^C \cong A^{B\times C}$). Because of the existence of this natural arithmetic, cartesian closed categories have many applications in different branches of mathematics and informatics. For example, cartesian closed categories play an especially important role in logic because it is well known that

$$
\begin{tikzcd}
\mathbf{B^A \times A} \arrow[r, "\mathbf{ev}"] & \mathbf{B} \\
\mathbf{C \times A} \arrow[u, dashed, "\mathbf{g \times id_A}"] \arrow[ur, "\mathbf{f}"'] &
\end{tikzcd}
$$

Figure 9: Exponential diagram.

they form models of the so-called typed lambda-calculus, which is an important formal programming language.

Theorem 3.2 *The category* **Clo** *is cartesian closed with exponential objects given by* $\mathcal{C}_2^{\mathcal{C}_1} = [\mathcal{C}_1, \mathcal{C}_2]$ *for all closure systems* $\mathcal{C}_1,\ \mathcal{C}_2$.

Proof: We have seen that **Clo** has (not only) finite products given by direct products. Let $\mathcal{C}_1,\ \mathcal{C}_2$ be closure systems and let $[\mathcal{C}_1, \mathcal{C}_2] \times \mathcal{C}_1 \xrightarrow{ev} \mathcal{C}_2$ be the transformation given by $(h \uplus Y).ev = Y.h$ whenever $h \subseteq |[\mathcal{C}_1, \mathcal{C}_2]|$ (*i.e.*, $\mathcal{C}_1 \xrightarrow{h} \mathcal{C}_2$ is a transformation) and $Y \subseteq \mathcal{C}_2$. If h is a closed transformation and Y is a closed subset, then $Y.h$ is closed too, which means that ev is a closed transformation. Let $\mathcal{D}$ be a closure system and $\mathcal{D} \times \mathcal{C}_\infty \xrightarrow{\{} \mathcal{C}_\in$ be a closed transformation. Let $\mathcal{D} \xrightarrow{\}} [\mathcal{C}_\infty, \mathcal{C}_\in]$ be the transformation given by $Y.(Z.g) = (Z \uplus Y).f$ for all $Z \subseteq |\mathcal{D}|$ and all $Y \subseteq |\mathcal{C}_1|$. Then $(Z \uplus Y).(g \times id_{\mathcal{C}_1}).ev = (Z.g \uplus Y).ev = Y.(Z.g) = (Z \uplus Y).f$. Thus, we have found a unique transformation g for which the diagram from the definition of a cartesian closed category commutes. To complete the proof, we are to show that g is closed. To this account, let $Z \subseteq |\mathcal{D}|$ be a closed subset. Then, whenever $Y \subseteq |\mathcal{C}_1|$ is a closed subset, also $Y.(Z.g) \subseteq |\mathcal{C}_2|$ is closed because $Y.(Z.g) = (Z \uplus Y).f$ where f is closed. $\square$

References

[1] J. Adámek, Horst Herrlich, and George E. Strecker. *Abstract and Concrete Categories.* Wiley, New York, 1990.

[2] Mark A. Aizerman and Andrew V. Malishevski. General Theory of Best Variants Choice: Some Aspects. *IEEE Trans. on Automatic Control*, AC-26(5):1030–1040, Oct. 1981.

[3] A. A. Albert, editor. *Studies in Modern Algebra.* Math. Assoc. of Amer., Englewood Cliffs, NJ, 1963.

[4] Kenneth J. Arrow. Rational Choice Functions and Orderings. *Economica*, 26(102):121–127, May 1959.

[5] Graham Brightwell, H. Fay Dowker, Raquel S. Garcia, Joe Hensen, and Rafael D. Sorkin. 'Observables' in Causal Set Cosmology. *Phys.Rev.*, D67(084031), 2003.

[6] Nathalie Caspard and Bernard Monjardet. The lattices of closure systems, closure operators and implicational systems on a finite set: A survey. *Discrete Applied Math.*, 127(2):241–269, 2003.

[7] V. Claes, E. Lowen-Colebunders, and G. Sonck. Cartesian closed topological hull of the construct of closure spaces. *Theory and Appl. of Cat.*, 18:481–489, 2001.

[8] G. B. Dantzig. *Linear Programming and Extensions.* Princeton Univ. Press, Princeton, NJ, 1963.

[9] Klaus Denecke, Marcel Erné, and Shelly L. Wismath. *Galois Connections and Applications.* Kluwer Academic, Dordrecht, 2004.

[10] D. Dikranjan, E. Giuli, and A. Tozzi. Topological categories and closure operators. *Quaest. Math.*, 11:323–337, 1988.

[11] Paul H. Edelman. Abstract Convexity and Meet-distributive Lattices. In *Combinatorics and Ordered Sets*, pages 127–150, Arcata, CA, 1986.

[12] Paul H. Edelman and Robert E. Jamison. The Theory of Convex Geometries. *Geometriae Dedicata*, 19(3):247–270, Dec. 1985.

[13] C. J. Everett. Closure Operators and Galois Theory in Lattices. *Trans. Amer. Math. Soc.*, 55:493–513, 1944.

[14] Bernhard Ganter and Rudolf Wille. *Formal Concept Analysis - Mathematical Foundations.* Springer Verlag, Heidelberg, 1999.

[15] Robert E. Jamison and John L. Pfaltz. Closure Spaces that are not Uniquely Generated. *Discrete Appl Math.*, 147:69–79, Feb. 2005. also in Ordinal and Symbolic Data Analysis, OSDA 2000, Brussels, Belgium July 2000.

[16] Robert E. Jamison-Waldner. A Perspective on Abstract Convexity: Classifying Alignments by Varieties. In David C. Kay and Marilyn Breen, editors, *Convexity and Related Combinatorial Geometry*, New York, 1982. Marcel Dekker, Inc.

[17] Mark R. Johnson and Richard A. Dean. Locally complete path independent choice functions and their lattices. *Mathematical Social Sciences*, 42(1):53–87, 2001.

[18] Achim Jung. *Cartesian closed categories of domains.* Center for Mathematics and Computer Science, Amsterdam, Neth., 1989.

[19] Bernhard Korte, László Lovász, and Rainer Schrader. *Greedoids.* Springer-Verlag, Berlin, 1991.

[20] Gleb A. Koshevoy. Choice functions and abstract convex geometries. *Mathematical Social Sciences*, 38(1):35–44, 1999.

[21] Christian Lindig and Gregor Snelting. Assessing Modular Structure of Legacy Code Based on Mathematical Concept Analysis. In *Proc. of 1997 International Conf. on Software Engineering*, pages 349–359, Boston, MA, May 1997.

[22] A. V. Malishevski. Path Independence in Serial-parallel Data Processing. *Mathematical Social Sciences*, 27(3):335–367, June 1994.

[23] Bernard Monjardet. A Use for Frequently Rediscovering a Concept. *Order*, 1:415–416, 1985.

[24] Bernard Monjardet. Closure operators and choice operators: a survey. In *Fifth Intern. Conf. on Concept Lattices and their Applications*, Montpellier, France, Oct. 2007. Lecture notes.

[25] Bernard Monjardet and V. Raderinirina. The duality between the antiexchange closure operators and the path independent choice operators on a finite set. *Math. Social Sciences*, 41(2):131–150, 2001.

[26] Oystein Ore. Galois Connexions. *Trans. of AMS*, 55:493–513, 1944.

[27] Oystein Ore. Mappings of Closure Relations. *Annals of Math.*, 47(1):56–72, Jan. 1946.

[28] Nicolas Pasquier, Yves Bastide, Rafik Taouil, and Lofti Lakhal. Closed Set Based Discovery of Small Covers for Association Rules. *Actes des 15e journes Bases de donnes avances*, pages 361–381, Oct. 1999.

[29] John L. Pfaltz. Convexity in Directed Graphs. *J. of Comb. Theory*, 10(2):143–162, Apr. 1971.

[30] John L. Pfaltz. Closure Lattices. *Discrete Mathematics*, 154:217–236, 1996.

[31] John L. Pfaltz. Establishing Logical Rules from Empirical Data. *Intern. Journal on Artificial Intelligence Tools*, 17(5):985–1001, Oct. 2008.

[32] John L. Pfaltz. Mathematical Continuity in Dynamic Social Networks. In Anwitaman Datta, Stuart Shulman, Baihua Zheng, Shoude Lin, , Aixin Sun, and Ee-Peng Lim, editors, *Third International Conf. on Social Informatics, SocInfo2011*, volume LNCS # 6984, pages 36–50, 2011.

[33] John L. Pfaltz and Josef Šlapal. Neighborhood Transformations. In *40th Southeastern International Conf. on Combinatorics, Graph Theory and Computing*, Boca Raton, FL, March 2009.

[34] Benjamin C. Pierce. *Basic Category Theory for Computer Scientists*. MIT Press, Cambridge, MA, 1991.

[35] Charles R. Plott. Path Independence, Rationality and Social Choice. *Econometrica*, 41(6):1075–1091, Nov. 1973.

[36] D. P. Rideout and Rafael D. Sorkin. Evidence for a Continuum Limit in Causal Set Dynamics. *Phys.Rev.*, D63(104011), 2001.

[37] Azriel Rosenfeld and John L. Pfaltz. Sequential Operations in Digital Picture Processing. *J. of the ACM*, 13(4):471–494, Oct. 1966.

[38] William C. Rounds and Guo-Qiang Zhang. Clausal Logic and Logic Programming in Algebraic Domains. *Information & Control*, 171:183–200, 2001.

[39] Amartya Sen. Social Choice Theory: A Re-Examination. *Econometricia*, 45(1):53–88, Jan. 1977.

[40] Rafael D. Sorkin. Causal Sets: Discrete Gravity. Prepared for School on Quantum Gravity, Valdivia, Chile, Jan. 2002.

[41] Harold S. Stone. *Discrete Mathematical Structures and Their Applications.* SRA, Palo Alto, 1973.

[42] Petko Valtchev, Rokia Missaoui, and Robert Godin. A Framework for Incremental Generation of Frequent Closed Itemsets. In Peter Hammer, editor, *Workshop on Discrete Mathematics & Data Mining, 2nd SIAM Conf. on Data Mining*, pages 75–86, Arlington, VA, April 2002.

[43] Josef Šlapal. A Galois Correspondence for Digital Topology . In K. Denecke, M. Erné, and S. L. Wismath, editors, *Galois Connections and Applications*, pages 413–424. Kluwer Academic, Dordrecht, 2004.

[44] Rudolf Wille. Restructuring Lattice Theory: An approach based on hierarchies of concepts. In Ivan Rival, editor, *Ordered Sets*, pages 445–470. Reidel, 1982.